Fast algorithms for Vizing's theorem on bounded degree graphs (Anton Bernshteyn and Abhishek Dhawan; 2023)

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1 Definitions

A chain of length k is a sequence of edges $C = (e_0, \ldots, e_{k-1})$ such that e_i and e_{i+1} are adjacent for every $0 \leq i < k-1$. For a proper partial edge coloring φ of G, we define a new partial edge coloring $\mathsf{Shift}(\varphi, C)$ (assuming that $\varphi(e_0) = \sqcup$ and $\varphi(e_i) \neq \sqcup$ for $0 < i < k$) as follows:

$$\begin{aligned} \mathsf{Shift}_0(\varphi, C) &:= \varphi \\ \mathsf{Shift}_{i+1}(\varphi, C) &:= \mathsf{Shift}(\mathsf{Shift}_i(\varphi), e_i, e_{i+1}) \text{ for } 0 \leq i < k-1 \\ \mathsf{Shift}(\varphi, C) &:= \mathsf{Shift}_{k-1}(\varphi, C), \text{where} \end{aligned} \qquad \mathsf{Shift}(\varphi, f, h)(e) := \begin{cases} \varphi(h) & e = f \\ \sqcup & e = h \\ \varphi(e) & \text{otherwise.} \end{cases}$$

For colors α and β, let $G[\alpha\beta]$ be the spanning subgraph of G containing only edges colored α or β.

Definition 1. We say that a chain C is φ-happy if it is φ-shiftable and $\mathsf{End}(C)$ is a $\mathsf{Shift}(\varphi, C)$-happy edge.

Definition 2. A chain $P = (e_0, \ldots, e_{k-1})$ is a path chain if $(e_1, \ldots, e_{k-1})$ is a path.

- For $\alpha \in M(\varphi, x)$ and $\beta \in M(\varphi, y)$, $P(xy; \varphi, \alpha\beta) := (xy, e_1, \cdots, e_{k-1})$ is a path chain, where $(e_1, \cdots, e_{k-1})$ is the maximal path in $G[\alpha\beta]$ starting at y.

Definition 3. A fan is a chain of the form $F = (xy_0, \ldots, xy_{k-1})$, where x is called the pivot of F. If F is φ-shiftable and not φ-happy, then we say that

- F is $(\varphi, \alpha\beta)$-hopeful if $\mathsf{deg}(x) < 2$ and $\mathsf{deg}(y_0) < 2$ in $G[\alpha\beta]$.
- F is $(\varphi, \alpha\beta)$-successful if F is $(\varphi, \alpha\beta)$-hopeful and x and y_0 are not connected in $G[\alpha\beta]$ under $\mathsf{Shift}(\varphi, F)$.

Definition 4. A Vizing chain in a proper partial edge coloring φ is a chain of the form $F + P$, where F is a $(\varphi, \alpha\beta)$-hopeful fan for some colors α, β and P is an initial segment of the path chain $P(\mathsf{End}(F); \mathsf{Shift}(\varphi, F), \alpha\beta)$ with $\mathsf{vStart}(P) = \mathsf{Pivot}(F)$.

Definition 5. A k-step Vizing chain is a chain of the form $C = C_0 + \ldots + C_{k-1}$, where $C_i = F_i + P_i$ is a Vizing chain in $\mathsf{Shift}(\varphi, C_0 + \ldots + C_{i-1})$ for all $0 \leq i < k-1$.

Definition 6. A k-step Vizing chain $C = C_0 + \cdots + C_{k-1}$, where $C_i = F_i + P_i$, is non-intersecting if, for all $0 \leq i < j < k$,

$$V(F_i) \cap V(F_j + P_j) = \emptyset \text{ and } E_{\mathsf{int}}(P_i) \cap E(E_j + P_j) = \emptyset.$$

2 The algorithms

Algorithm F: Coloring edges of a given graph G with $\Delta + 1$ colors

Input : A graph G with maximum degree Δ.
Output: A proper $(\Delta+1)$-edge coloring φ of G.

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φ ← ∅, U ← E(G)
while U ≠ ∅ do
    Pick an edge e ∈ U and a vertex x ∈ e uniformly at random
    Compute a φ-happy multi-step Vizing Chain C by running Algorithm M with the input (G, φ, e, x)
    Augment φ using C
    U ← U \ {e}
return φ
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Theorem 1. Given a graph G with maximum degree Δ, Algorithm F computes a proper $(\Delta+1)$-edge coloring (assuming that F terminates with G).

Algorithm M (sketch): Computing φ-happy multi-step Vizing chain

Input : A graph G, a proper partial edge coloring φ of G, an uncolored edge $e = xy$, and a vertex $x \in e$.

Output: A fan F with $\mathsf{Start}(F) = e$, $\mathsf{Pivot}(F) = x$ and a path P with $\mathsf{Start}(P) = \mathsf{End}(F)$, $\mathsf{vStart}(P) = \mathsf{Pivot}(F) = x$.

1 $(F, P) \leftarrow \mathsf{FirstChain}(\varphi, xy, x)$
2 $C \leftarrow (xy)$, $\psi \leftarrow \varphi$, $k \leftarrow 0$
3 **while** true **do**
4   **if** $\mathsf{length}(P) < 2\ell$ **then**
5   **return** $C + F + P$
6   Choose $\ell' \in [\ell, 2\ell - 1]$ uniformly at random, $F_k \leftarrow F$, $P_k \leftarrow P|\ell'$
7   Let α, β be such that P_k is an $\alpha\beta$-path where $\psi(\mathsf{End}(P_k)) = \beta$
8     $\psi \leftarrow \mathsf{Shift}(\psi, F_k + P_k)$
9     $uv \leftarrow \mathsf{End}(P_k)$, $v \leftarrow \mathsf{vEnd}(P_k)$, $(\hat{F}, \hat{P}) \leftarrow \mathsf{NextChain}(\psi, uv, u, \alpha, \beta)$
10   **if** $C + F_k + P_k + \hat{F} + \hat{P}$ is intersecting **then**
11   Let j be the index such that the first intersection occurs at $F_j + P_j$
12   Restore to the step where F_j and P_j were constructed
13         $F \leftarrow F_j$, $P \leftarrow P_j|2\ell$
14   **else if** $2 \leq \mathsf{length}(\hat{P}) < 2\ell$ **and** $\mathsf{vEnd}(\hat{P}) = \mathsf{Pivot}(\hat{F})$ **then**
15   **return** FAIL
16   **else**
17         $C \leftarrow C + F_k + P_k$, $F \leftarrow \hat{F}$, $P \leftarrow \hat{P}$, $k \leftarrow k + 1$
18 **return** φ

3 Time complexity

Theorem 2 (Main theorem)**.** Let G be a graph, $n := |V(G)|$, and $\Delta := \Delta(G) \geq 2$. Algorithm F outputs a proper $(\Delta + 1)$-edge coloring of G in time $\mathsf{poly}(\Delta)n$ with probability at least $1 - 1/\Delta^n$.

The Main theorem follows (with some additional work) from the following theorem:

Theorem 3. Let $e = xy$ be an uncolored edge. For $t > 0$ and $\ell \geq 1200\Delta^{16}$, Algorithm M with input (G, φ, xy, x) outputs a φ-happy multi-step Vizing chain of length $O(\ell t)$ in time $O(\Delta \ell t)$ with probability at least $1 - 4m(1200\Delta^{15}\ell)^{t/2}$.

We fix a graph G and a partial $(\Delta + 1)$-edge coloring of G. The first t iterations of Algorithm M are uniquely determined by the input sequence $(f, z, \ell_1, \ldots, \ell_t)$, where $f \in E(G)$ is an uncolored edge, $z \in f$, and $\ell_i \in [\ell, 2\ell - 1]$ (ℓ_i is the random choice made at step 6 in the i-th iteration).

Definition 7. Let $\mathcal{I}^{(t)}$ be the set of all input sequences for which Algorithm M does not terminate within t iterations.

Definition 8. The record of $I \in \mathcal{I}^{(t)}$ is a tuple $D(I) = (d_1, \ldots, d_t)$, where d_i is computed at the i-th of Algorithm M with the input sequence I as follows

$$d_i := \begin{cases} 1 & \text{if we reach step 17} \\ j - k & \text{if we reach step 11.} \end{cases}$$

The terminus of I is the pair $\tau(I) = (\mathsf{End}(C), \mathsf{vEnd}(C))$.

Definition 9. Let $\mathcal{D}^{(t)}$ be the set of all tuples D such that $D = D(I)$ for some $I \in \mathcal{I}^{(t)}$. Given $D \in \mathcal{D}^{(t)}$ and a pair (uv, u) such that $uv \in E(G)$, we let $\mathcal{I}^{(t)}(D, uv, u)$ be the set of all input sequences $I \in \mathcal{I}^{(t)}$ such that $D(I) = D$ and $\tau(I) = (uv, u)$.

Definition 10. $\mathcal{D}_s^{(t)} := \{D = (d_1, \ldots, d_t) \in \mathcal{D}^t \ : \ \sum_{i=1}^t d_i = s\}$

Lemma 1. Let $D \in \mathcal{D}^{(t)}$ and $uv \in E$. Then $|\mathcal{I}^{(t)}(D, uv, u)| \leq \mathsf{wt}(D)$, where $\mathsf{wt}(D)$ is some suitable funcion.

Lemma 2. Let $D \in \mathcal{D}_s^{(t)}$. Then $\mathsf{wt}(D) \leq (75\Delta^{15}\ell)^{t/2}(75\Delta^7\ell)^{-s/2}$.

Lemma 3. $|\mathcal{D}_s^{(t)}| \leq 4^t$.