Rooting algebraic vertices of convergent sequences

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Convergence of graphs

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- Let G be an infinite graph, can we approximate its properties by finite graphs?

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- Function $f, g, h, \ldots$

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Some properties cannot be expressed. For example

- "The graph is connected."
- "The graph contains a Hamiltonian path."

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Definition

Let G be a finite graph and ϕ a first-order with $p \geq 0$ free variables, i.e. $\phi \in \mathrm{FO}_p$. We define the *Stone pairing* of ϕ and G to be

$$\langle \phi, G \rangle = \frac{|\phi(G)|}{|V(G)|^p},$$

where $\phi(G) = \{\mathbf{v} \in V(G)^p : G \models \phi(\mathbf{v})\}$ is the solution set of ϕ in G.

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Definition

A sequence (G_n) of finite graphs is FO-*convergent* if the sequence $(\langle \phi, G_n \rangle)$ converges for each first-order formula ϕ in the language of graphs.

Limit structure

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Graph L on a (nice) probability space $(V(L), \Sigma_L, \nu_L)$ with the property that $\phi(L) \in \Sigma_L^p$ for each $\phi \in \mathrm{FO}_p$ is called a *modeling*. For a modeling L and a formula $\phi \in \mathrm{FO}_p$, we define their Stone pairing as

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Definition

We say that a modeling L is an FO-*limit* of an FO-convergent sequence (G_n) if for each $\phi \in \mathrm{FO}$ we have

$$\lim_{n \to \infty} \langle \phi, G_n \rangle = \langle \phi, L \rangle.$$

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Theorem (Trakhtenbrot, 1950)

Given an a sentence ϕ, it is undecidable whether there exists a finite graph G satisfying ϕ.

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The game lasts for k rounds, each consists of:

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Duplicator wins if $\{a_i \mapsto b_i\}$ is an isomorphism between $G[a_1, \ldots, a_k]$ and $H[b_1, \ldots, b_k]$.

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For graphs G, H the following are equivalent:

1. *G and H are indistinguishable by sentences of q-rank k,*
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- If $G_n = G(n, p)$ for fixed p, then (G_n) is almost surely FO-convergent. A modeling limit does not exists.
- If

$$G_n = \begin{cases} G(n,p) & n \text{ odd}, \\ G(n,q) & n \text{ even}. \end{cases}$$

  for fixed $p < q$, then G_n is almost surely not FO-convergent.

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Example (Homomorphism convergence)

Consider a finite graph F on $[|V(F)|]$. Let $\phi_F(x_1, \ldots, x_{|V(F)|})$ be the formula $\bigwedge_{ij \in E(F)} x_i \sim x_j$. Then for any finite graph G we have

$$t(F, G) = \langle \phi_F, G \rangle,$$

where $t(F, G)$ is the homomorphism density of F in G.

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Example (Benjamini-Schramm convergence)

Consider a finite graph F rooted at vertex o. Let $\phi_{(F,o)}(x)$ be the formula expressing "the neighborhood of x is isomorphic to (F, o)". Then for any finite graph G we have

$$\rho((F, o), G) = \langle \phi_{(F,o)}, G \rangle,$$

where $\rho((F, o), G)$ is the "density of balls (F, o)" in G.

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Definition

The language of rooted graphs consists of the adjacency relation $\sim$ and the constant 'Root'. The symbol FO^+ stands for the set of formulas in the language of rooted graphs.

Question (Nešetřil, Ossona de Mendez)

Suppose that a modeling L is an FO*-limit of a sequence* (G_n)*. Let* r *be a vertex of* L*. Is it true that there are vertices* $r_n \in V(G_n)$ *such that* (L, r) *is the* FO*-limit of the sequence* $((G_n, r_n))$*?*

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Theorem (Christofides, Král')

- *There is an example of (G_n), L, and r such that the required sequence (r_n) does not exists.*

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Theorem (Christofides, Kráľ)

1. *There is an example of (G_n), L, and r such that the required sequence (r_n) does not exists.*
2. *If the root r is selected at random (using ν_L), the sequence (r_n) exists almost surely.*

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A formula $\phi \in \mathrm{FO}_1$ is called *algebraic* in a graph G if the solution set $\phi(G)$ is finite. A vertex $v \in V(G)$ is *algebraic* in G if there is an algebraic formula $\phi \in \mathrm{FO}_1$ in G such that $G \models \phi(v)$.

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Suppose that a modeling L is an FO-limit of a sequence (G_n) and r is an algebraic vertex of L. Then there exist vertices $r_n \in V(G_n)$ such that (L, r) is an FO-limit of the sequence $((G_n, r_n))$.

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This is tight: if r contained in just a *countable* definable set, the sequence (r_n) needs not to exist.

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Suppose that a modeling L is an FO-limit of a sequence (G_n) and ξ is an algebraic formula in L. Then there exist vertices $r_n \in \xi(G_n)$ and $r \in \xi(L)$ such that (L, r) is an FO-limit of the sequence $((G_n, r_n))$.

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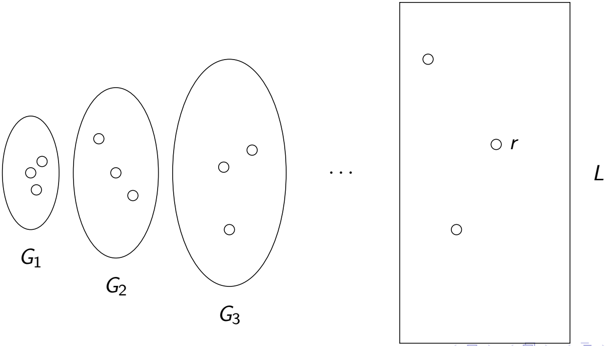

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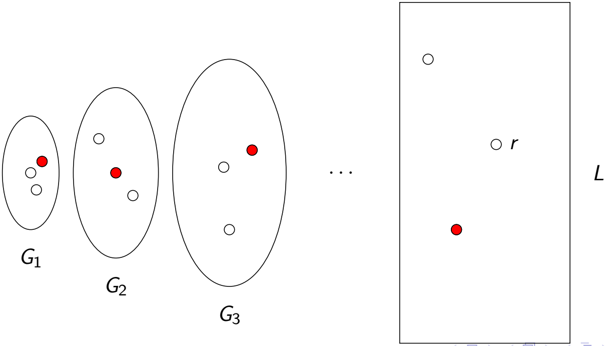

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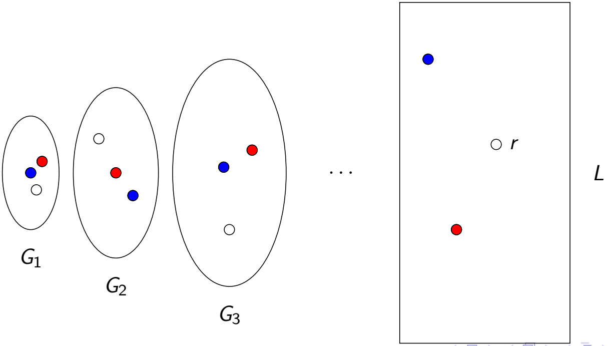

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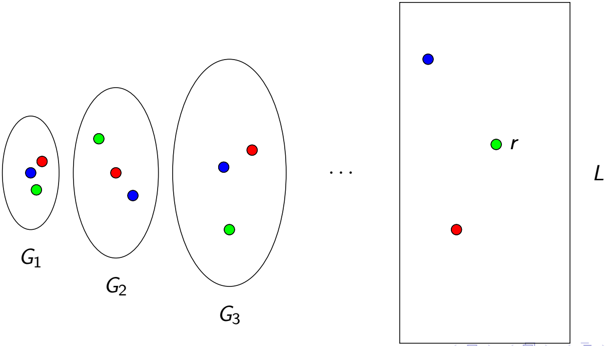

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There exist roots $r_n \in \xi(G_n), r \in \xi(L)$ such that for any $\phi \in \mathrm{FO}^+$ we have

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For any $\phi_1, \ldots, \phi_k \in \mathrm{FO}^{+}$ there exist roots $r_n \in \xi(G_n), r \in \xi(L)$ such that for each $i \in [k]$ we have

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Then use a compactness argument to prove Theorem 2.

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Idea

Take $r_n \in \xi(G_n)$, resp. $r \in \xi(L)$, that minimize $\langle \phi, (G_n, r_n) \rangle$. If Lemma 1 holds, this has to work.

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Theorem (Girard-Newton formulas)

The coefficients of the polynomial $p(x) = \prod_{i=1}^{n}(x - a_i)$ can be obtained by basic arithmetic operations from values $z_1, \ldots, z_n$, where $z_k = \sum_{i=1}^{n} a_i^k$.

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We show that

$$\sum_{u \in \xi(G_n)} \langle \phi, (G_n, u_i) \rangle^k = \langle \psi_k, G \rangle$$

for some formula $\psi_k \in \mathrm{FO}$, $k \in [|\xi(G_n)|]$.

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Replace the constant Root in $\phi(\boldsymbol{x}) \in \mathrm{FO}_p^+$ by a new free variable y to obtain $\phi^-(\boldsymbol{x}, y) \in \mathrm{FO}_{p+1}$.

Proof of Lemma 1, continuation

We use formulas $\psi_{k,\ell}(\boldsymbol{x})$ defined as follows:

$$(\exists y_1, \ldots, y_\ell) \left(\bigwedge_{i=1}^{\ell} \xi(y_i) \wedge \bigwedge_{1 \leq i < j \leq \ell} y_i \neq y_j \wedge \bigwedge_{i=1}^{k} \bigwedge_{j=1}^{\ell} \phi^{-}(\boldsymbol{x}_i, y_j) \right)$$

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Proposition

Fix $0 < p < q < 1$. The sequence

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is almost surely FO*-convergent and admits a modeling limit L whose smaller part A_L is countable.*

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There is no sequence of vertices $r_n \in A_n$ such that the sequence (G_n, r_n) even converges. In particular, (L, r) for $r \in A_L$ is not a limit of (G_n, r_n).

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- Thus, the question FO-convergence reduces to QF-convergence $\Leftrightarrow$ homomorphism convergence.
- The sequence clearly converges.

Construction of a limit

- There is a construction of Goldstern, Grossberg, and Kojman[1] of a homogeneous bipartite graph with parts $A = \omega$ and $B \subseteq \{\text{infinite sequences of natural numbers}\}$ where $|B| = 2^\omega$.

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- Which is not difficult.

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Concluding remarks

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- Can we decide about the other vertices?
- Can we decide about set of vertices of measure > 0?

Thank you.

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Questions?