Handout: Tiling edge-ordered graphs

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Abstract

Given graphs F and G, a perfect F-tiling in G is a collection of vertex-disjoint copies of F in G that together cover all the vertices in G. The study of the minimum degree threshold forcing a perfect F-tiling in a graph G has a long history, culminating in the Kühn–Osthus theorem which resolves this problem, up to an additive constant, for all graphs F.

In this paper the authors initiate the study of the analogous question for edge-ordered graphs. An *edge-ordered graph* G is a graph equipped with a total order $\leq$ of its edge set $E(G)$. In particular, Araujo et. al. characterize edge-ordered graphs F for which this problem is well-defined.

1 Introduction

Definition 1.1 (Turánable)**.** An edge-ordered graph F is *Turánable* if there exists a $t \in \mathbb{N}$ such every edge-ordering of the graph K_t contains a copy of F.

Gerbner, Methuku, Nagy, Pálvölgyi, Tardos, and Vizer initiated a systematic study of Turán problem for edge-ordered graphs. In particular, they proved the following result:

Theorem 1.2 (Turánable characterization)**.** *An edge-ordered graph F on f vertices is Turánable if and only if all four canonical edge-orderings of K_f contain a copy of F.*

Definition 1.3. Given $n \in \mathbb{N}$, we denote by $\{v_1, \dots, v_n\}$ the vertex set of the complete graph K_n. The following labelings L_1, L_2, L_3, and L_4 induce the *canonical orderings* of K_n.

- *min ordering*: For $1 \leq i < j \leq n$ the label of the edge v_iv_j is $L_1(v_iv_j) = 2ni + j - 1$.
- *max ordering*: For $1 \leq i < j \leq n$ the label of the edge v_iv_j is $L_2(v_iv_j) = (2n-1)j + i$.
- *inverse min ordering*: For $1 \leq i < j \leq n$ the label of the edge v_iv_j is $L_3(v_iv_j) = (2n+1)i - j$.
- *inverse max ordering*: For $1 \leq i < j \leq n$ the label of the edge v_iv_j is $L_4(v_iv_j) = 2nj - i + n$.

2 Main result

Definition 2.1 (Tileable)**.** An edge-ordered graph F on f vertices is *tileable* if there exists a $t \in \mathbb{N}$ divisible by f such that every edge-ordering of the graph K_t contains a perfect F-tiling.

Theorem 2.2 (Tileable characterization)**.** *An edge-ordered graph F on f vertices is tileable if and only if all twenty $\star$-canonical orderings of K_f contain a copy of F.*

Definition 2.3. Let $\{x, v_1, \dots, v_n\}$ denote the vertex set of K_{n+1}. Suppose $L : E(K_{n+1}) \to \mathbb{R}$ is a labeling of the edges of K_{n+1} such that its restriction to $K_{n+1} - x$ is canonical with one of the standard labelings L_1, L_2, L_3, or L_4. Moreover, suppose that the labels $x_i := L(xv_i)$ for $i \in [n]$ satisfy one of the following:

- *Larger increasing orderings*: $x_n > \cdots > x_2 > x_1 > \max_{i<j}\{L(v_iv_j)\}$.
- *Larger decreasing orderings*: $x_1 > x_2 > \cdots > x_n > \max_{i<j}\{L(v_iv_j)\}$.
- *Smaller increasing orderings*: $x_1 < x_2 < \cdots < x_n < \min_{i<j}\{L(v_iv_j)\}$.
- *Smaller decreasing orderings*: $x_n < \cdots < x_2 < x_1 < \min_{i<j}\{L(v_iv_j)\}$.
- *Middle increasing orderings*: $x_i = 2ni$ for all $i \in [n]$.

Then, L induces a *$\star$-canonical ordering* of K_{n+1}. We refer to the vertex x as *the special vertex*.

Proposition 2.4. *Consider the edge-ordered graph D_n defined as a graph on vertices $u_1, \dots, u_n$ containing all edges incident to u_1 or u_n. The edges are ordered as $u_1u_2 < u_1u_3 < \cdots < u_1u_n < u_2u_n < \cdots < u_{n-1}u_n$. Let $n \geq 4$. Then D_n is Turánable but is not tileable.*

Lemma 2.5. *An edge-ordered graph F is tileable if and only if there exists an $n \in \mathbb{N}$ such that the following holds. Every edge-ordering of K_n such that $K_n - x$ is canonical for some vertex $x \in V(K_n)$ contains a copy of F that covers x.*

3 Concluding remarks

For the characterization of Turánable graphs, namely Theorem 1.2, all four canonical orderings are necessary in the following sense: for every $n \geq 4$ and every canonical ordering $K_n^{\leq}$ of K_n, there is a non-Turánable edge-ordered n-vertex graph F such that F can be embedded into all the canonical orderings of K_n other than $K_n^{\leq}$. Thus, it is natural to raise the following question.

Question 3.1. *Are all twenty $\star$-canonical orderings necessary in Theorem 2.2? That is, does Theorem 2.2 still hold if we omit some of the $\star$-canonical orderings from the statement?*