No-three-in-line-problem on a torus

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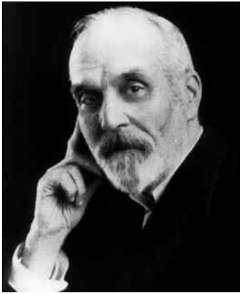
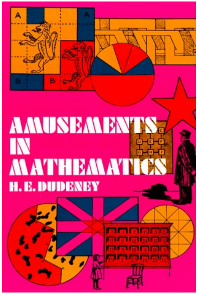




Puzzle 317

Place two pawns in the middle of the chessboard, one at Q4 and the other at K5. Now, place the remaining fourteen pawns (sixteen in all) so that no three shall be in a straight line in any possible direction.

No-three-in-line-problem

How many points can be placed on an $n \times n$ grid so that no three points are collinear.

- Still unsolved for general n.

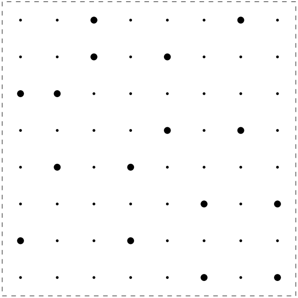


Figure: Dudeney's solution for the chessboard (8×8 grid).

Discrete torus $T_{m\times n}$

Cartesian product $\{0,\ldots,m-1\}\times\{0,\ldots,n-1\}\subset\mathbb{Z}^2$.

Line on $T_{m\times n}$ is an image of a line in $\mathbb{Z}^2$ under a mapping which maps a point $(x,y)\in\mathbb{Z}^2$ to the point $(x \mod m, y \mod n)$.

Line in $\mathbb{Z}^2$ $\{(b_1,b_2)+k(v_1,v_2); k\in\mathbb{Z}\}$, where $\gcd(v_1,v_2)=1$.

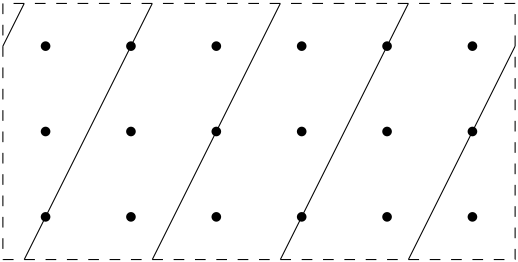
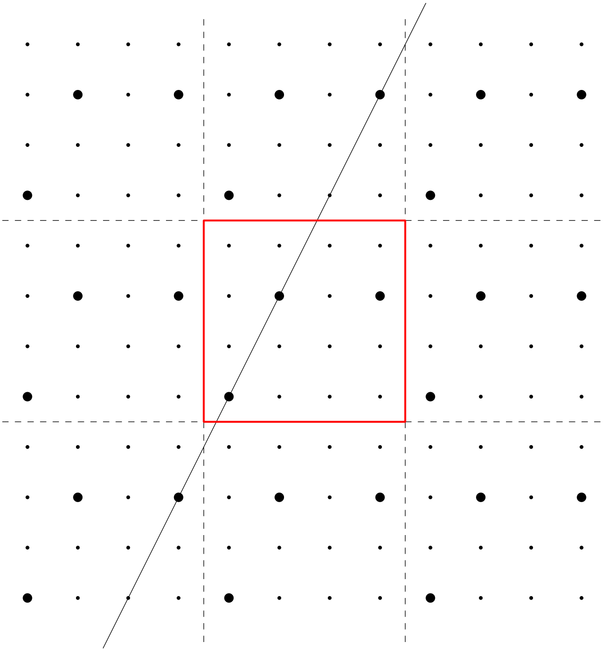
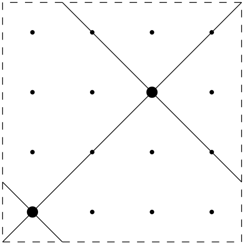


Figure: $T_{3\times 6}$

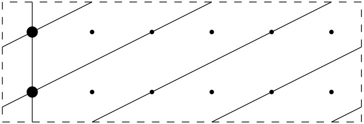


Discrete torus $T_{m\times n}$

- More lines between two points.



- A line is a proper subset of another line.



No-three-in-line-problem on a torus

No-three-in-line-problem on a torus [Fowler at al. 2012]

How many points can be placed on a discrete torus $T_{m \times n}$ of size $m \times n$ so that no three points are collinear.

- Let $\tau_{m,n}$ denote such maximum number of points.

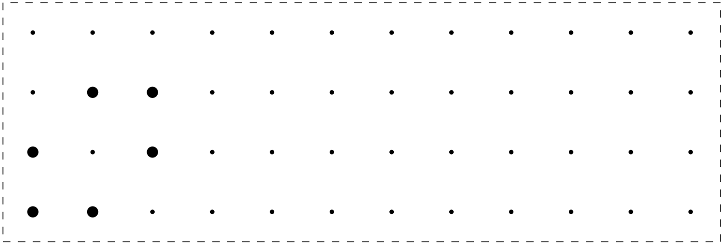


Figure: $\tau_{4,12} = 6$.

Algebraic viewpoint

$T_{m\times n}$ is an abelian group $\mathbb{Z}_m \times \mathbb{Z}_n$.

Line on $T_{m\times n}$ is a coset of a cyclic subgroup.

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Question

What is $\tau_{m,n}$ for coprime m, n?

Algebraic viewpoint

$T_{m \times n}$ is an abelian group $\mathbb{Z}_m \times \mathbb{Z}_n$.

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Question

What is $\tau_{m,n}$ for coprime m, n?

- $\tau_{m,n} = 2$ by the Chinese remainder theorem.

Known results

- $\tau_{m,n} = 2$ if $\gcd(m,n) = 1$. [Misiak, Stępień, A. Szymaszkiewicz, L. Szymaszkiewicz, Zwierzchowski]
- $\tau_{m,n} \leq 2\gcd(m,n)$. [Misiak et al.]
- $\tau_{m,n} \leq \tau_{xm,yn}$. [Misiak et al.]
- $\tau_{m,n} = \tau_{xm,yn}$ if $\gcd(x,y) = \gcd(m,y) = \gcd(n,x) = 1$. [MS, Misiak et al. for prime $m = n$]

- $\tau_{p,p} = p + 1$. [Fowler, Groot, Pandya, Snapp]
- $\tau_{p^a,p^{(a-1)p+2}} = 2p^a$. [MS, Misiak et al. for $a = 1$]
- $\tau_{2^a,2^{2a-1}} = 2^{a+1}$. [MS]
- $\tau_{p^a,p^a} \leq p^a + p^{\lceil \frac{a}{2} \rceil} + 1$. [MS]

- The sequence $\tau_{z,1}, \tau_{z,2}, \tau_{z,3}, \ldots$ is periodic for all z. [MS]

Sequences

If we fix one coordinate of a torus, we get the sequence $\tau_{z,1}, \tau_{z,2}, \tau_{z,3}, \ldots$ for $z \geq 2$, which we denote σ_z.

z	1	2	3	4	5	6	7	8	9	10	11	12	13	...
2	2	4	2	4	2	4	2	4	2	4	2	4	2	...
3	2	2	4	2	2	4	2	2	6	2	2	4	2	...
4	2	4	2	6	2	4	2	8	2	4	2	6	2	...
5	2	2	2	2	6	2	2	2	2	6	2	2	2	...
6	2	4	4	4	2	8	2	4	6	4	2	8	2	...

Table: Initial values of $\tau_{z,n}$.

- The potential maximum of the sequence is $2z$. Since $\tau_{m,n} \leq 2\gcd(m, n)$.