No-three-in-line problem on a torus: periodicity

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Basic definitions

- *Discrete torus* $T_{m\times n}$ of size $m \times n$ is a set $\{0,\dots,m-1\} \times \{0,\dots,n-1\} \subset \mathbb{Z}^2$.
- *Line on* $T_{m\times n}$ is an image of a line in $\mathbb{Z}^2$ under a mapping which maps a point $(x,y) \in \mathbb{Z}^2$ to the point $(x \mod m, y \mod n)$.
- *Line in* $\mathbb{Z}^2$ $\{(b_1,b_2) + k(v_1,v_2); k \in \mathbb{Z}\}$, where $\gcd(v_1,v_2) = 1$.

Notions

- $\tau_{m,n}$ denotes the maximal number of points which can be placed on the discrete torus of sizes $m \times n$ so that no three of these points are collinear.
- σ_z is a sequence which we obtain by fixing one of the coordinates of a torus. In other words $\sigma_z(x) := \tau_{z,x}$.
- $\pi_{m,n}$ denotes the mapping which maps a point $(x,y) \in \mathbb{Z}^2$ to the point $(x \mod m, y \mod n)$.

Main results

Theorem 1. *The sequence σ_z is periodic for all positive integers z greater than 1.*

When z is a power of a prime we can say more:

Theorem 2. *Let $T_{p^a \times p^{(a-1)p+2}}$ be a torus where p is a prime and $a \in \mathbb{N}$. Then $\tau_{p^a, p^{(a-1)p+2}} = 2p^a$.*

Theorem 3. *Let p be a prime, $a \in \mathbb{N}$. Let us denote $m := \min\{x; \sigma_{p^a}(x) = 2p^a\}$. Then $m = p^b$ for some $b \geq a$ and the sequence σ_{p^a} is periodic with the period m.*

Tools

Theorem 4 (Chinese Remainder Theorem)**.** *Let m, n be positive integers. Then two simultaneous congruences*

$$x \equiv a \pmod{m},$$
$$x \equiv b \pmod{n}$$

are solvable if and only if $a \equiv b \pmod{\gcd(m,n)}$. Moreover, the solution is unique modulo $\operatorname{lcm}(m,n)$, where lcm *denotes the least common multiple.*

Theorem 5 (Dirichlet's Theorem)**.** *Let a, b be positive relatively prime integers. Then there are infinitely many primes of the form $a + nb$, where n is a non-negative integer.*

Theorem 6 (Langrange's Theorem)**.** *Let G be a group and H its subgroup. Then $|G| = [G\colon H] \cdot |H|$.*

Theorem 7. *Let $m, n \in \mathbb{N}$. Then $\tau_{m,n} \leq 2\gcd(m, n)$.*

Lemma 8. *Let m, n, x, y be positive integers. Then $\tau_{m,n} \leq \tau_{xm,yn}$.*

Lemma 9. *Let m, n, x, y be positive integers such that m, n are not both 1 and $\gcd(x, y) = \gcd(m, y) = \gcd(n, x) = 1$. Then $\tau_{m,n} = \tau_{xm,yn}$.*

Lemma 10. *Let $z \in \mathbb{N}$ and $z = \prod_{i \in I} p_i^{a_i}$ be its prime factorization. There exists $m_z = \prod_{i \in I} p_i^b$, where $b \geq a_i$ for each $i \in I$ which satisfies the following condition.*

$$\forall J \subseteq I : \sigma_z \left(\prod_{i \in \overline{J}} p_i^b \prod_{i \in J} p_i^{c_i} \right) = \sigma_z \left(\prod_{i \in \overline{J}} p_i^{d_i} \prod_{i \in J} p_i^{c_i} \right)$$

for arbitrary $0 \leq c_i < b$, $d_i \geq b$ and where $\overline{J} := I \setminus J$.

Other known results

- $\tau_{p,p} = p + 1$.
- $\tau_{2^a,2^{2a-1}} = 2^{a+1}$.
- $\tau_{p^a,p^a} \leq p^a + p^{\lceil \frac{a}{2} \rceil} + 1$.