Shorter Labeling Schemes for Planar Graphs

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Definition 1. Let $\mathcal{C}$ be a class of graphs. An *adjacency labeling scheme* for $\mathcal{C}$ is a pair $\langle \lambda, \varphi \rangle$ of functions such that, for every graph $G \in \mathcal{C}$, it holds:

- λ is the *Encoder* that assigns to every vertex u of G a different binary string $\lambda(u, G)$; and
- φ is the *Decoder* that decides adjacency from the labels taken from G. More precisely, for every pair u, v of vertices of G, $\varphi(\lambda(u,G), \lambda(v,G))$ is TRUE if and only if u, v are adjacent in G.

The *length* of the labeling scheme $\langle \lambda, \varphi \rangle$ is the function $\ell \colon \mathbb{N} \to \mathbb{N}$ that maps every $n \in \mathbb{N}$ to the maximum length, expressed in the number of bits, of labels assigned by the Encoder in n-vertex graphs from $\mathcal{C}$.

1 Planar Graphs

Theorem 2. *The class of connected planar graphs with n vertices admits a labeling scheme of length $\log n + \log d + O(\log \log n)$, where d is the radius of the graph. The Encoder runs in polynomial time and the Decoder in constant time.*

Moreover, if the graph is provided together with a vertex subset Q, then the Encoder may assign to the vertices of Q labels of length at most $\log |Q| + \log d + O(\log \log n)$.

Theorem 3. *Planar graphs with n vertices admit a labeling scheme of length $\frac{4}{3} \log n + O(\log \log n)$. The Encoder runs in polynomial time and the Decoder in constant time.*

2 Bounded Treewidth Graphs

Definition 4. A *tree decomposition* of a graph G is a pair (T, β), where $\beta \colon V(T) \to 2^{V(G)}$ assigns a *bag* $\beta(v)$ to each node v of T, such that

- for every vertex p of G, there exists $v \in V(T)$ such that $p \in \beta(v)$,
- for every edge pq in G, there exists $v \in V(T)$ such that $p, q \in \beta(v)$, and
- for every vertex p of G, the set $T_p = \{v \in V(T) : p \in \beta(v)\}$ induces a connected subtree of T.

The *width* of the tree decomposition is the maximum of $|\beta(v)| - 1$ over all $v \in V(T)$. The *tree-width* of a graph G is the minimum width of a tree decomposition of G.

Definition 5. A *bidecomposition* of a graph H is a pair (T, α), where T is a binary rooted tree and α maps vertices H to nodes of T, so that for every edge uv of H, $\alpha(u)$ and $\alpha(v)$ are related.

Lemma 6 (Gavoille, Labourel)**.** *Let G be an n-vertex graph of treewidth at most k. Then there exists a bidecomposition (T, α) of G satisfying the following:*
(A1) $|\alpha^{-1}(x)| = O(k \log n)$ for every node x of T; and
(A2) T has depth at most $\log n$.
Moreover, for every fixed k, given G such a bidecomposition can be constructed in time $O(n \log n)$.

Lemma 7. *Let G be an n-vertex graph of treewidth at most k and $S \subseteq V(G)$. Then there exists a bidecomposition (T, α) of G satisfying the following:*
(B1) $|\alpha^{-1}(x)| = O(k \log n)$ for every node x of T;
(B2) T has depth at most $\log n + O(1)$; and
(B3) for every $u \in S$, $\alpha(u)$ is at depth at most $\log |S| + O(1)$ in T.
Moreover, for every fixed k, given G and S such a decomposition can be constructed in polynomial time, with the degree of the polynomial independent of k.

Theorem 8. *For any fixed $k \in \mathbb{N}$, the class of graphs of treewidth at most k admits a labeling scheme $\langle \lambda, \varphi \rangle$ of length $\log n + O(k \log \log n)$ with the following properties:*
(P1) From any label a one can extract in time $O(1)$ an identifier $\iota(a)$, so that the Decoder may be implemented as follows: given a label a, one may compute in time $O(k)$ a set $\Gamma(a)$ consisting of at most k identifiers so that $\varphi(a, b)$ is TRUE *if and only if $\iota(a) \in \Gamma(b)$ or $\iota(b) \in \Gamma(a)$.*
(P2) If the input graph G is given together with a vertex subset Q, then the scheme can assign to the vertices of Q labels of length $\log |Q| + O(k \log \log n)$.
The Encoder works in polynomial time while the Decoder works in constant time.

3 Tools

Definition 9. Let $\mathcal{P}$ be a partition of the vertex set of a graph G. The *quotient graph $G/\mathcal{P}$* has $\mathcal{P}$ as its vertex set, and two different parts $A, B \in \mathcal{P}$ are considered adjacent in $G/\mathcal{P}$ if and only if there exists $a \in A$ and $b \in B$ such that a and b are adjacent in G.

Theorem 10 (Dujmović et al.)**.** *Let G be a planar graph, and let F be any BFS forest of G. Then, one can construct in polynomial time a partition $\mathcal{P}$ of the vertex set of G such that every part of $\mathcal{P}$ is the vertex set of a column of F and the quotient graph $G/\mathcal{P}$ has treewidth at most 8.*

Lemma 11. *Every connected planar graph of radius at most ρ has treewidth at most 3ρ.*

Theorem 12 (Steinitz lemma)**.** *Let $x_1, \ldots, x_n \in \mathbb{R}^m$ such that*

$$\sum_{i=1}^{n} x_i = 0 \quad \text{and} \quad \|x_i\| \leqslant 1 \text{ for each } i.$$

There exists a permutation $\pi \in S_n$ such that all partial sums satisfy

$$\|\sum_{j=1}^{k} x_{\pi(j)}\| \leqslant m \text{ for all } k = 1, \ldots, n.$$