1 First part. Towards the proof of the Orbit conjecture.

1.1 Definitions and notation.

P is a subset of $\mathbb{R}^2$ of size n. No three points lie on a line and no four points lie on a circle.

T is a **triangulation**, i.e. a maximal subset of $\binom{P}{2}$ with no crossing edges.

Edge flip. Let T triangulation $ABCD$ empty convex of P, $e \in T$ between AD. Then change it to CB and you obtain a new traingulation.

The **flip graph** of P is a graph whose vertices are triangulations of P and there is an edge between two triangulations if you can obtain one from the other by an edge flip.

Delanuay triangulation is the triangulation whose angle vector is maximal.

$\mathcal{T}$ is a **labelled triangulation**, i.e a pair (T, l) where T is a triangulation and $l : T \to \{1, \ldots, t_p\}$ a bijection. The number t_p is the total amount of edges in T.

Two edges e and f are in the same orbit if there exists a labelled triangulation $\mathcal{T}$ and a sequencce of edge flips from $\mathcal{T}$ to some $\mathcal{T}'$ such that $l(e) = l'(f)$.

An elementary swap of e and f is a sequence of edge flips $\sigma\pi\sigma^{-1}$, where σ takes $l(e)$ and $l(f)$ to the diagonals of a pentagon and π is the permutation of these diagonals insine the pentagon.

1.2 Storyline

Theorem 1 (Lawson 1971) *The flip graph is connected.*

Theorem 2 (Orbit theorem) *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two labelled traingulations, then there exist a flip sequence from $\mathcal{T}_1$ to $\mathcal{T}_2$ if and only if for all edges $e_1 \in \mathcal{T}_1$ and $e_2 \in \mathcal{T}_2$ such that $l_1(e_1) = l(e_2)$ are in the same orbit. Furthermore, there is a polynomial time algorithm that tests wheter the condition is satisfied, and if it is, computes a flip sequence of length $O(n^7)$ to transform $\mathcal{T}_1$ to $\mathcal{T}_2$.*

Theorem 3 (Elementary swap theorem) *Given a labelled triangulation $\mathcal{T}$, any permutation of the labels that can be realized by a sequence of edge flips can be realized by a sequence of elementary swaps.*

Theorem 4 *There exists a cell complex X with the following properties: (1) The 1-skeleton of X is the flip graph of P. (2) The 2-cells of X are in bijection with 4 and 5 elementary cycles of P. (3) The fundamental group of X is trivial.*

Theorem 5 *Let $\mathcal{T}$ be a labeled triangulation, two edges are in the same orbit if and only if there exist an elementary swap between them.*

Theorem 6 (Edge label permutation theorem) *Let T be a triangulation of a point set P in the plane, l_1 and l_2 two labelings of the edges of T. Then, there is a sequence of $O(n)$ elementary swaps from l_1 to l_2 if and only if all $e, e' \in T$ such that $l_1(e) = l_2(e')$ are in the same orbit. Such a sequence can be realized via a sequence of $O(n^7)$ edge flips, which can be found in polynomial time.*

2 Second Part. Towards the proof of Theorem 4

2.1 Definitions and notations.

The **complex of plane graphs on P** is defined by $T = T(P) = \{F : F \subset E, F \text{ non-crossing}\}$.

A pure d-dimensional simplicial complex is **shellable** if its facets admit an order such that at each step the simplicial complex generated by the previous facets intersected with the current one is a (d-1) pure simplicial complex.

Two polyhedra $|K|$ and $|L|$ are **piecewise linear homeomorphic** if exist isomorphic subdivisions of K and L, i.e exists a face preserving bijection.

A simplicial complex K is a **combinatorial n-manifold** if the link of each p-simplex is piecewise linear homeomorphic to either the boundary of an (n-p)-simplex or to an (n-p-1)-simplex.

Let K be a combinatorial n-manifold and K' its first barycentric subdivision. Given a p-simplex σ in K, $K'|lk(\sigma, K)$ is isomorphic to the subcomplex $\tilde{K}_\sigma = \{\tilde{\tau}_1 \dots \tilde{\tau}_m : \sigma < \tau_1 < \dots < \tau_m \in K, \sigma \neq \tau_1\}$ in K'. Thus $|\tilde{K}_\sigma| \simeq S^{n-p-1}$ or B^{n-p-1}, and hence $\tilde{B}_\sigma = \tilde{\sigma} * |\tilde{K}_\sigma|$ is a piecewise linear (n-p)-ball. $\tilde{B}_\sigma$ is the **dual cell** of σ and the collection of dual cells is the **dual cell complex**.

We say that a topological space X is a **n-dimensional pseudomanifold with boundary** if: (a) $X = |K|$ is the union of all n-simplices. (b) Every (n–1)-simplex is a face of one or two n-simplices for $n > 1$. (c) For every pair of n-simplices σ and σ' in K, there is a sequence of n-simplices $\sigma = \sigma_0, \sigma_1, \dots, \sigma_k = \sigma'$ such that the intersection $\sigma_i \cap \sigma_{i+1}$ is an (n-1)-simplex for all $i = 0, \dots, k-1$. The boundary of X is given by faces which are contained in a unique maximal face.

2.2 Storyline

Theorem 7 ([1] Prop 4.7.22) *Suppose K is a finite d-dimensional simplicial complex that is a pseudomanifold i.e. K is pure and every (d-1) dimensional face of K is contained in at most two d-faces. If K is shellable then K is either a piecewise linear ball or a piecewise-linear sphere. The former case occurs if and only if there is at least one (d-1) dimensional face that is contained in only one d-face of K, in which case the pseudomanifold is said to have nonempty boundary.*

Theorem 8 *T is shellable $(m-1)$ dimensional pseudomanifold with nonempty boundary, and hence a piecewise linear ball.*

Theorem 9 *Let T be the simplicial complex of plane graphs on the point set P. A non-crossing set of edges F on P is an interior face of T if and only the following conditions holds: 1) F contains all convex hull edges of P. 2) Every bounded region in the complement of the plane graph (P, F) is convex.*

Theorem 10 *Let B be a d-dimensional piecewise linear ball. 1) For each interior k-dimensional face F of B, one can define a dual cell F^*. [2][Lemma I.19] 2) The construction reverses inclusion. 3) The dual cells of the interior faces of B form a regular cell complex, denoted B^* and called the dual complex. B^* need not be a manifold or pure d dimensional, but it is homotopy equivalent to B. [3][Lemma 70.1]*

Reference

[1] A. Bjorner, A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. M. Ziegler, *Oriented matroids.* No. 46, Cambridge University Press, 1999.

[2] J. Johnson, "Notes on piecewise-linear topology," 2018.

[3] J. R. Munkres, *Elements of algebraic topology.* CRC Press, 2018.