DISTRIBUTED COLORING IN SPARSE GRAPHS WITH FEWER COLORS

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1. Definitions, main theorem

chromatic number $\chi(G)$,

family of lists $(L(v))_{v\in G}$ is *k-list-assignment* if $|L(v)| \geqslant k\ \forall v \in G$,

G is *L-list-colorable*, if $\forall v \in G$, $c(v) \in L(v)$.

G is *k-list-colorable*, if for any k-list-assignement L, the graph G is L-list-colorable,

choice number of G, $\mathrm{ch}(G)$, is the minimum integer k such that G is k-list-colorable,

average degree of G is the average of the degrees of the vertices of G,

maximum average degree of G, $\mathrm{mad}(G)$, is the max of the avg degrees of the subgraphs of G,

$\chi(G) \leqslant \mathrm{ch}(G) \leqslant \lfloor \mathrm{mad}(G) \rfloor + 1$,

arboricity of G, $a(G)$, is the min nbr of edge-disjoint forests into which the edges of G can be partitioned. $a(G) = \max\left\{ \left\lceil \frac{|E(H)|}{|V(H)|-1} \right\rceil \mid H \subseteq G, |V(H)| \geqslant 2 \right\}$.

A *block* of a graph G is a maximal 2-connected subgraph of G.

A *Gallai tree* is a connected graph in which each block is an odd cycle or a clique.

Theorem (Brooks)**.** *Any connected graph of maximum degree Δ which is not an odd cycle or a clique has chromatic number at most Δ.*

Theorem 1.1 ([2, 3])**.** *If a connected graph G is not a Gallai tree, then for any list-assignment L such that for every vertex $v \in G$, $|L(v)| \geqslant d_G(v)$, G is L-list-colorable.*

Theorem 1.2 (Folklore)**.** *Let G be a graph and let $d = \lceil \mathrm{mad}(G) \rceil$. If $d \geqslant 3$ and G does not contain any $(d+1)$-clique, then $\chi(G) \leqslant ch(G) \leqslant d$.*

Theorem 1.3 (Main result)**.** *There is a deterministic distributed algorithm that given an n-vertex graph G, and an integer $d \geqslant \max(3, \mathrm{mad}(G))$, either finds a $(d+1)$-clique in G, or finds a d-list-coloring of G in $O(d^4 \log^3 n)$ rounds. Moreover, if every vertex has degree at most d, then the algorithm runs in $O(d^2 \log^3 n)$ rounds.*

Theorem 1.4. *No distributed algorithm can 4-color every n-vertex planar graph in $o(n)$ rounds.*

2. Proof of Theorem 1.3

Lemma 2.1. $|A| \geqslant \frac{n}{(3d)^3}$*. Moreover, if there are no poor vertices in G, then* $|A| \geqslant \frac{n}{12d+1}$.

Lemma 2.2. *Any L-list-coloring of $G-A$ can be extended to an L-list-coloring of G in $O(d \log^2 n)$ rounds.*

3. Proof of Lemmas

Theorem 3.1 ([1])**.** *If a graph G has girth at least g (g odd), and average degree $d = 2+\delta$, for some real number $\delta > 0$, then*

$$n \geqslant 1 + d \sum_{i=0}^{\frac{g-1}{2}} (d-1)^i \geqslant (1+\delta)^{\frac{g-1}{2}}.$$

Corollary 3.2. *If an n-vertex graph G has girth at least g, and average degree at least $2+\delta$, for some real number $\delta > 0$, then*

$$g \leqslant \tfrac{4}{\log(1+\delta)} \log n.$$

Observation 3.3. *If three vertices u, v, w of a maximal clique K are in a local block of $G[S]$, then K is a local block of $G[S]$.*

Proposition 3.4. *There are at least $\frac{1}{12}|S|$ vertices of degree at most $d-1$ in $G[S]$.*

Observation 3.5. *For any vertex $v \in H$, $|L_H(v)| \geqslant d - d_{G'}(v) + d_H(v)$. In particular, if $d_{G'}(v) \leqslant d$ then $|L_H(v)| \geqslant d_H(v)$ and if $d_{G'}(v) \leqslant d-1$ then $|L_H(v)| \geqslant d_H(v)+1$.*

4. Consequences of main result

Corollary 4.1. *There is a deterministic distributed algorithm of round complexity $O(\Delta^2 \log^3 n)$ that given any n-vertex graph of maximum degree $\Delta \geqslant 3$, and any Δ-list-assignment L for the vertices of G, either finds an L-list-coloring of G, or finds that no such coloring exists.*

Proposition 4.2. *Every n-vertex planar graph of girth at least g has maximum average degree less than $\frac{2g}{g-2}$. In particular, planar graphs have maximum average degree less than 6, triangle-free planar graphs have maximum average degree less than 4, and planar graphs of girth at least 6 have maximum average degree less than 3.*

Corollary 4.3. *There is a deterministic distributed algorithm of round complexity $O(\log^3 n)$ that given an n-vertex planar graph G,*

(1) *finds a 6-(list-)coloring of G;*
(2) *finds a 4-(list-)coloring of G if G is triangle-free;*
(3) *finds a 3-(list-)coloring of G if G has girth at least 6.*

Observation 4.4. *Let G be a graph, and H be a graph with at most $|V(G)|$ vertices, such that each ball of radius at most r in H is isomorphic to some ball of radius at most r in G. Then no distributed algorithm can color G with less than $\chi(H)$ colors in at most r rounds.*

Theorem 4.5. *No distributed algorithm can 3-color the graph H_k in less than $k/2$ rounds. In particular, no distributed algorithm can 3-color every planar triangle-free graph on n vertices in $o(n)$ rounds.*

Theorem 4.6. *No distributed algorithm can 3-color the rectangular $k \times k$-grid in the plane in less than $k/2$ rounds. In particular, no distributed algorithm can 3-color every planar bipartite graph on n vertices in $o(\sqrt{n})$ rounds.*

Corollary 4.7. *For any integer $g \geqslant 1$, there is a deterministic distributed algorithm of round complexity $O(\log^3 n)$ that given an n-vertex graph G embeddable on a surface of Euler genus g, finds an $H(g)$-list-coloring of G. Moreover, when $\frac{1}{2}(5+\sqrt{24g+1})$ is an integer and G is not the complete graph on $H(g)$ vertices, the algorithm can indeed find an $(H(g)-1)$-list-coloring of G.*

5. Conclusion

Theorem 5.1. *There is a deterministic distributed algorithm that given an n-vertex graph G of maximum degree Δ, and a nice list-assignment L for the vertices of G, finds an L-list-coloring of G in $O(\Delta^2 \log^3 n)$ rounds.*

References

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