Linear programming. Given a set H of n hyperplanes in $\mathbb{R}^d$, find a point p which lies on or below all hyperplanes in H, while minimizing a given linear objective function.

Basic definitions.

Definition 1. Given $p \in \mathbb{R}^d$, let $\text{Violate}_p(H) = \{h \in H;\ h \text{ is strictly below } p\}$. A subset $R \subseteq H$ is an *ε-net* of H if for every $p \in \mathbb{R}^d$,

$$\text{Violate}_p(R) = \emptyset \implies |\text{Violate}_p(H)| \le \varepsilon |H|.$$

Definition 2. Let $\rho_p(H) = |\text{Violate}_p(H)|/|H|$. A subset $R \subseteq H$ is a *sensitive ε-approximation of* H with respect to $P \subseteq \mathbb{R}^d$, if for every $p \in P$,

$$|\rho_p(R) - \rho_p(H)| \le (\varepsilon/2)\sqrt{\rho_p(H)} + \varepsilon^2/2.$$

Main result. Deterministic algorithm for linear programming with running time:

- $\mathcal{O}(d^3 \log^c d)^d n$,
- $\mathcal{O}(d)^{(2+\mathcal{O}(\delta))d} n$ for any constant $\delta > 0$,
- $\mathcal{O}(d)^{d/2} (\log d)^{3d} n$ with n a power of 2.

Tools

Lemma 1.

(a) (Mergeability) *If R_i is an ε-net of H_i for each i, then $\bigcup_i R_i$ is an ε-net of $\bigcup_i H_i$.*

(b) *Given a set H of $n \ge d$ hyperplanes in $\mathbb{R}^d$, we can construct an ε-net of H of size $\mathcal{O}(\frac{d}{\varepsilon} \log n)$ in $\mathcal{O}(n/d)^{d+\mathcal{O}(1)}$ deterministic time.*

Fact 2. *Define* $\text{Cap}(H, \alpha) = \{p \in \mathbb{R}^d; \rho_p(H) \le \alpha\}$.

(a) (Mergeability) *If R_i is a sensitive ε-approximation of H_i w.r.t P and the H_i have equal size, then $\bigcup_i R_i$ is a sensitive ε-approximation of H w.r.t. P.*

(b) (Reducibility) *If R is a sensitive ε-approximation of H w.r.t. P and R' is a sensitive ε'-approximation of R w.r.t. P, then R' is a sensitive $(\varepsilon + 2\varepsilon')$ approximation of H w.r.t. P.*

(c) *Given a set H of n hyperplanes in $\mathbb{R}^d$ with $n \ge r \ge \frac{c_0 d}{\varepsilon^2} \log n$ for some constant c_0, and given a parameter $t \ge 1$, we can construct a sensitive ε-approximation of H w.r.t.* $\text{Cap}(H, (t\varepsilon)^2)$ *of size r in $\mathcal{O}(t\varepsilon n/d)^{d+\mathcal{O}(1)}$ deterministic time.*

Lemma 3.

(a) *If R is a sensitive $\sqrt{\varepsilon}$-approximation of H w.r.t.* $\text{Cap}(H, \varepsilon + \frac{1}{|H|})$*, then R is an ε-net of H.*

(b) *If R is a sensitive $c\varepsilon$-approximation of H w.r.t.* $\text{Cap}(H, (t\varepsilon)^2)$*, then* $\text{Cap}(H, (t\varepsilon)^2) \subseteq \text{Cap}(R, ((t+c)\varepsilon)^2)$.