Definition. A *parity game* is a game on a directed graph on n vertices with vertices numbered by numbers $\{1, \dots, m\}$ (for $m \leq n$). The players Anke and Boris take turns in moving a pebble from a vertex to vertex along arcs (there is at least one possible move from each vertex). Anke and Boris have prescribed parities and the winner is the player of the same parity as limes superior of the numbers visited by the pebble.

Theorem 1. *There exists an alternating polylogarithmic space algorithm deciding which player has a winning strategy in a given parity game. When the game has n nodes and the values of the nodes are in the set $\{1, 2, 3, \dots, m\}$, then the algorithm runs in $\mathcal{O}(\log(n)\log(m))$ alternating space.*

Proof.

Definition. In Anke's winning statistics, an i-sequence is a sequence of nodes $a_1, a_2, a_3, \dots, a_{2^i}$ which have been visited (not necessarily consecutively, but in order) during the play so far such that, for each $k \in \{1, 2, 3, \dots, 2^{i-1}\}$,

$$\max\{val(a) \mid a = a_k \vee a = a_{k+1} \vee \text{a was visited between } a_k \text{ and } a_{k+1}\},$$

is of Anke's parity.

Definition (Invariants)**.** We use the following invariants:

1. Only b_i with $0 \leq i \leq \lceil \log(n) \rceil + 2$ are considered and each such b_i is either zero or a value of a node which occurs in the play so far.
2. An entry b_i refers to an i-sequence which occurred in the play so far iff $b_i > 0$.
3. If b_i, b_j are both non-zero and $i < j$ then $b_i \leq b_j$.
4. If b_i, b_j are both non-zero and $i < j$, then in the play of the game so far, the i-sequence starts only after a node with value b_j was visited at or after the end of the j-sequence.

Definition (Algorithm)**.** We use this algorithm to update values of b_is:

1. If b is of Anke's parity or $b > b_i > 0$ for some i, then one selects the largest i such that
   (a) either b_i is not of Anke's parity – that is, it is either 0 or of Boris' parity – but all b_j with $j < i$ and also b are of Anke's parity
   (b) or $0 < b_i < b$

   and then one updates $b_i = b$ and $b_j = 0$ for all $j < i$.
2. If this update produces a non-zero b_i for any i with $2^i > 2n$ then the play terminates with Anke being declared winner.

□

Theorem 2. *There is an algorithm which finds the winner of a parity game with n nodes and values from $\{1, 2, 3, \dots, m\}$ in time $\mathcal{O}(n^{c\log(m)})$.*

Theorem 3. *If $m \leq \log(n)$ then one can solve the parity game with n nodes having values from $\{1, 2, 3, \dots, m\}$ in time $\mathcal{O}(n^5)$.*