Flag Algebras: A First Glance

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Fix a family $\mathcal{H}$ of forbidden subgraphs.

Definition 1 (Type)**.** *A* type *of* size k *is an $\mathcal{H}$-free graph σ with $V(\sigma) = [k]$. The empty type is denoted by $\varnothing$.*

Definition 2 (Flag)**.** *A* σ-flag *is a pair (F, θ) where F is an $\mathcal{H}$-free graph and θ is an embedding of σ into F. When the embedding itself is not important, we will drop it, speaking simply of the σ-flag F.*

For $n \geq |\sigma|$, denote by $\mathcal{F}_n^\sigma$ the set of all σ-flags of size n, taken up to isomorphism; denote by $\mathcal{F}^\sigma$ the set of all σ-flags taken up to isomorphism. A type σ is *degenerate* if $\mathcal{F}^\sigma$ is finite. We assume that all types are nondegenerate.

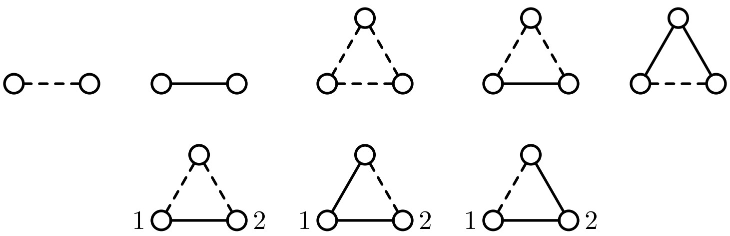

Figure 1: Let $\mathcal{H} = \{\triangle\}$. On the top row we have all $\varnothing$-flags of sizes 2 and 3, up to isomorphism (nonedges are shown as dashed lines); notice that the triangle itself is not a flag. On the bottom row we have all flags of type σ = 1∘—∘2; notice that the last two of these flags are not isomorphic, since the isomorphism has to preserve the labels.

We say that σ-flags $F_1, \ldots, F_t$ *fit* in a σ-flag G if

$$|G| - |\sigma| \geq (|F_1| - |\sigma|) + \cdots + (|F_t| - |\sigma|).$$

Definition 3 (Density)**.** *Let $F_1, \ldots, F_t$ and (G, θ) be σ-flags such that $F_1, \ldots, F_t$ fit in G. Choose pairwise-disjoint sets $U_1, \ldots, U_t \subseteq V(G) \setminus \operatorname{Im}\theta$ uniformly at random. The* density *of $F_1, \ldots, F_t$ in G, denoted $p(F_1, \ldots, F_t; G)$, is the probability that the σ-flag $(G[U_i \cup \operatorname{Im}\theta], \theta)$ is isomorphic to F_i for $i = 1, \ldots, t$.*

Theorem 1 (Chain rule)**.** *If $F_1, \ldots, F_t$, and G are σ-flags such that $F_1, \ldots, F_t$ fit in G, then for every $1 \leq s \leq t$ and every n such that $F_1, \ldots, F_s$ fit in a σ-flag of size n and a σ-flag of size n together with $F_{s+1}, \ldots, F_t$ fit in G, the identity*

$$p(F_1, \ldots, F_t; G) = \sum_{F \in \mathcal{F}_n^\sigma} p(F_1, \ldots, F_s; F) p(F, F_{s+1}, \ldots, F_t; G)$$

holds.

Theorem 2. *If F_1, F_2 are fixed σ-flags, then there exists a function $f(n) = O(1/n)$ such that if F_1, F_2 fit in a σ-flag G, then $|p(F_1, F_2; G) - p(F_1; G)p(F_2; G)| \leq f(|G|)$.*

Definition 4 (Limit functional)**.** *Let $(A_k)_{k\geq 0}$ be a convergent sequence in $\mathcal{F}^\sigma$ and let $\phi(F) = \lim_{k\to\infty} p(F; A_k)$ be the pointwise limit of the functions $p(\,\cdot\,; A_k)$. Extend ϕ linearly to $\mathbb{R}\mathcal{F}^\sigma$, obtaining a linear functional. We say that ϕ is the* limit functional *of the convergent sequence $(A_k)_{k\geq 0}$ or, when the sequence itself is not relevant, that it is a* limit functional.