The list chromatic number of graphs with small clique number

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1 Introduction

- Set of colors C.
- Each vertex v of a graph G has a list of color $L_v \subseteq C, |L_v| = q$.
- We are looking for proper coloring $\varphi : V(G) \to C$ such that for every vertex v: $\varphi(v) \in L_v$.
- The list chromatic number of G, denoting by $\chi_\ell(G)$, is the minimum q such that for any list assignment exists coloring which respects the lists.

Theorem 1. *For every $\varepsilon > 0$ there exists Δ_ε such that every triangle-free graph G with maximum degree $\Delta \geq \Delta_\varepsilon$ has $\chi_\ell(G) \leq (1+\varepsilon)\Delta/\ln\Delta$.*

2 Tools

Theorem 2 (Lovász Local Lemma)**.** *Let $A_1, \ldots, A_n$ be a set of random events, each with probability at most $1/4$. For each $i \in [n]$ we have a subset $\mathcal{D}_i$ of the events such that A_i is mutually independent of all other event outside of $\mathcal{D}_i$. If for each $i \in [n]$ we have $\sum_{j\in\mathcal{D}_i} \Pr[A_j] < 1/4$ then $\Pr[\bar{A}_1 \cap \cdots \cap \bar{A}_n] > 0$.*

Notation

- N_v – neighborhood of v.
- $F_v \subseteq L_v$ – free colors for v with Blank.
- $D_{v,c} = \{u \in N_v | \varphi(u) = \text{ Blank}, c \in F_u\}$ – candidates for the color c.
- $\alpha = \Delta^{\varepsilon/2}$ – limit for available colors.
- Flaws of partial coloring φ:
  - $B_v : |F_v| < \alpha$.
  - $Z_v : \sum_{c\in F_v} |D_{v,c}| > \frac{\alpha}{10} \cdot |F_v|$.

Lemma 3. *Let $v \in V(G)$. We assign a random color from F_u to each vertex $u \in N_v$ (uniformly, independently). Then, $\Pr[B_v], \Pr[Z_v] < \Delta^{-4}$.*

Lemma 4. *Suppose we have a partial list coloring φ such that for every vertex v, neither B_v nor Z_v hold. Then, we can color the blank vertices to obtain a full list coloring.*

3 Algorithm

We fix an ordering $<$ of $V(G)$ and order the flaws:

$$\forall u, v \in V(G) : B_u \prec Z_v$$
$$\forall u, v \in V(G), u < v : B_u \prec B_v, Z_u \prec Z_v$$

Consider a partial coloring φ and any flaw f_v of φ. We try to remove the flaw f_v using randomized algorithm.

FIX(f_v, φ)

Set $\mathcal{L} = \{F_u : u \in N_v\}$.
Write "COLORS ℓ*" where* $f_v = \gamma(\mathcal{L}, \ell)$.

(*) $\forall u \in N_v : \varphi'(u) \in_r F_u$ ($\varphi'(w) = \varphi(w)$ for all other vertices).
While $\exists B_w, Z_w : \mathrm{dist}(w, v) \leq 2$:
Let g_w be the least such flaw and call $\varphi' = \mathrm{FIX}(g, \varphi')$. *Write "FIX(B,*$\ell$*)" or "FIX(Z,*$\ell$*)" where* $w = \beta(v, \ell)$.
Return φ'. *Write "Return".*

Observation 5. *Let* φ' *be a coloring returned by FIX*(f_v, φ)*:*

1. *The flaw* f_v *does not hold.*
2. *There are no flaws that did not hold in* φ*.*

More Notation

- φ_0 – an initially coloring.
- f – any flaw of φ_0.
- φ_t – current coloring after t execution of (*).
- H_t – log of FIX after t execution of (*).
- R_t – random bits used for all execution of (*).

Lemma 6. *We can reconstruct the first* t *steps of FIX from* $\varphi_0, \varphi_t, f_v$ *and* H_t*.*

Lemma 7. *For any partial coloring* φ *and any flaw* f_v *of* φ*, the probability that FIX*(f_v, φ) *continues for at least* n *executions of (*) is at most* $\Delta^{-n/2}$*.*

4 K_r-free Graphs

Theorem 8. *For any* $r \geq 4$*, every* K_r*-free graph* G *with maximum degree* Δ *has* $\chi_\ell(G) \leq 200r\frac{\Delta \ln \ln \Delta}{\ln \Delta}$.