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The Erdős discrepancy problem (Terence Tao)

Nonstandard Ramsey-Type Principles

Definition 1.[Discrepancy] Given a sequence $f\colon \mathbb{N} \to H$ taking values in a real or complex Hilbert space H, define the discrepancy of f to be the quantity

$$\sup_{n,d\in\mathbb{N}} \left\| \sum_{j=1}^{n} f(jd) \right\|_H$$

Theorem 1.1 (Erdős discrepancy problem, vector-valued case)**.** *Let H be a real or complex Hilbert space, and let $f\colon \mathbb{N} \to H$ be a function such that $\|f(n)\|_H = 1$ for each $n \in \mathbb{N}$. Then the discrepancy of f is infinite.*

Theorem 1.8 (Equivalent form of vector-valued Erdős discrepancy problem)**.** *Let $g\colon \mathbb{N} \to S^1$ be a stochastic completely multiplicative function taking values in the unit circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ (where we give the space $(S^1)^{\mathbb{N}}$ of functions from $\mathbb{N}$ to S^1 the product σ-algebra). Then*

$$\sup_n \mathbb{E} \left| \sum_{j=1}^{n} g(j) \right|^2 = +\infty$$

Theorem 1.9 (Measure-theoretic formulation)**.** *Let (Ω, μ) be a probability space, and let $g\colon \Omega \to (S^1)^{\mathbb{N}}$ be a measurable function to the space $(S^1)^{\mathbb{N}}$ of functions from $\mathbb{N}$ to S^1, such that $g(\omega) \in (S^1)^{\mathbb{N}}$ is completely multiplicative for μ-almost every $\omega \in \Omega$ (that is to say, $g(\omega)(nm) = g(\omega)(n)g(\omega)(m)$ for all $n, m \in \mathbb{N}$ and μ-almost all $\omega \in \Omega$). Then one has*

$$\sup_n \int_\Omega \left| \sum_{j=1}^{n} g(\omega)(j) \right|^2 d\mu(\omega) = +\infty$$

Proof that Theorem 1.1 implies Theorem 1.8

If (Ω, μ) and g are as in Theorem 1.9 one takes H to be the complex Hilbert space $L^2(\Omega, \mu)$ and for each natural number n we let $f(n) \in H$ be the function

$$f(n)\colon \omega \mapsto g(\omega)(n).$$

Proof that Theorem 1.9 implies Theorem 1.1

Observation 2. *We claim that it suffices to construct, for each $X \geq 1$, a stochastic completely multiplicative function g_x taking values in S^1 such that*

$$\mathbb{E} \left| \sum_{j=1}^{n} g_x(j) \right| \ll_C 1$$

for all $n \leq X$, where the implied constant is uniform in n and X, but we allow the underlying probability space defining the stochastic function g_X to depend on X.

Definition 3.[Fourier expansion]

$$F(x) = \sum_{\xi \in (\mathbb{Z}/M\mathbb{Z})^r} \hat{F}(\xi) e\left(\frac{x \cdot \xi}{M}\right)$$

$$\hat{F}(\xi) = \frac{1}{M^r} \sum_{x \in (\mathbb{Z}/M\mathbb{Z})^r} F(x) e\left(-\frac{x \cdot \xi}{M}\right)$$

van der Corput argument

Theorem 1.10 (Logarithmically averaged nonasymptotic Elliott conjecture (Tao))**.** *Let a_1, a_2 be natural numbers, and let b_1, b_2 be integers such that $a_1b_2 - a_2b_1 \neq 0$. Let $\varepsilon > 0$, and suppose that A is sufficiently large depending on $\varepsilon, a_1, a_2, b_1, b_2$. Let $x \geq w \geq A$, and let $g_1, g_2 \colon \mathbb{N} \to \mathbb{C}$ be multiplicative functions with $|g_1(n)|, |g_2(n)| \leq 1$ for all n, with g_1 "non-pretentious" in the sense that*

$$\sum_{p \leq x} \frac{1 - \Re g_1(p)\overline{\chi(p)}p^{-it}}{p} \geq A$$

for all Dirichlet characters χ of period at most A, and all real numbers t with $|t| \leq Ax$. Then

$$\left| \sum_{x/w < n \leq x} \frac{g_1(a_1n + b_1)g_2(a_2n + b_2)}{n} \right| \geq \varepsilon \log \omega$$

Proposition 1.11 (van der Corput argument)**.** *Suppose that $g \colon \mathbb{N} \to S^1$ is a stochastic completely multiplicative function, such that*

$$\mathbb{E}\left|\sum_{j=1}^{n} g(j)\right|^2 \leq C^2$$

for some finite $C > 0$ and all natural numbers n (thus, g would be a counterexample to Theorem 1.8). Let $\varepsilon > 0$, and suppose that X is sufficiently large depending on ε, C. Then with probability $1 - O(\varepsilon)$, one can find a (stochastic) Dirichlet character χ of period $q = O_{C,\varepsilon}(1)$ and a (stochastic) real number $t = O_{C,\varepsilon}(X)$ such that

$$\sum_{p \leq X} \frac{1 - \Re g(p)\overline{\chi(p)}p^{-it}}{p} \ll_{C,\varepsilon} 1$$