Distance Graphs and Sets of Positive Upper Density in $\mathbb{R}^d$

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1 Introduction

Let $A \subset \mathbb{R}^d$. The *u*pper Banach density of A is defined by

$$\delta^*(A) = \lim_{N\to\infty} \sup_{t\in\mathbb{R}^d} \frac{|A \cap (t + Q_N)|}{|Q_N|},$$

where $|\cdot|$ is a standard Lebesgue measure and $Q_N = [-N/2, N/2]^d$.

Theorem 1 *Let $\Delta_k \subset \mathbb{R}^d$ be a fixed collection of $k+1$ points in general position. If $A \subset \mathbb{R}^d$ has a positive upper Banach density and $d \geq k+1$, then there exist a threshold $\lambda_0 = \lambda_0(A, \Delta_k)$ such that A contains an isometric copy of $\lambda \cdot \Delta_k$ for all $\lambda \geq \lambda_0$.*

A *d*istance graph $\Gamma = \Gamma(V, E)$ is a connected finite graph with vertex set $V \subset \mathbb{R}^d$. Distance graph Γ is k-degenerate if each of its subgraphs contains a vertex with a degree at most k. In any k-degenerate distance graph Γ. We can order the vertices such that $V = \{v_0, v_1, \dots, v_n\}$ and

$$V_j := \{v_i : (v_i, v_j) \in E, 0 \leq i < j\}$$

satisfies $|V_j| \leq k$. A distance graph Γ is *p*roper if for ever $1 \leq j \leq n$, the set $V_j \cup v_j$ is in general position.

Theorem 2 *Let $\Gamma = \Gamma(V, E)$ be a proper k-degenerate distance graph and $d \geq k+1$.*

(i) If $A \subset \mathbb{R}^d$ has a positive upper Banach density, then there exist $\lambda_0 = \lambda_0(A, \Gamma)$ such that A contains an isometric copy of $\lambda \cdot \Gamma$ for all $\lambda \geq \lambda_0$.

(ii) If $A \subset [0,1]^d$ with $|A| > 0$, then A contains an isometric copy of $\lambda \cdot \Gamma$ for all λ in some interval of length at least $\exp(-C_\Gamma |A|^{C|V|})$.

2 Useful stuff

Let Γ be a fixed proper k-degenerate distance graph with vertex set $V = \{v_0, v_1, \dots, v_n\}$ and $k \leq d-1$. For each $(v_i, v_j) \in E$ we set $t_{ij} = |v_i - v_j|^2$. We define the configuration space

$$S_\Gamma := \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{d(n+1)} : x_0 = 0 \text{ and } |x_i - x_j|^2 = t_{ij} \text{ for all } (v_i, v_j) \in E\}.$$

Each of the spheres

$$S_j := \{x \in \mathbb{R}^d : |x - x_i|^2 = t_{ij} \text{ for all } x_i \in X_j\}$$

have dimension $d - |X_j|$.

For any $0 < \lambda \ll 1$ and functions $f_0, f_1, \dots, f_n : [0,1]^d \to \mathbb{R}$ with $d \geq k+1$ we define a *l*ocalized counting function as

$$T_\Gamma(f_0, f_1, \dots, f_n)(\lambda) := \iint \cdots \int f_0(x) f_1(x - \lambda x_1) \cdots f_n(x - \lambda x_n)\, d\mu_n(x_n) \cdots d\mu_1(x_1) dx.$$

Let $0 < L \ll 1$ and $f : [0,1]^d \to \mathbb{R}$. We define $U^1(L)$-norm as

$$||f||_{U^1(L)} = ||f \star \varphi_L||_2,$$

where $\varphi_L(x) = L^{-d}\varphi(L^{-1}x)$ and $\varphi = 1_{[-1/2,1/2]^d}$ is an indicator function.

Proposition 1 *Let $0 < \varepsilon, \lambda \ll 1$. For any $L < \varepsilon^6 \lambda$, $0 \leq m \leq n$ and functions $f_0, f_1, \dots, f_m : [0,1]^d \to [-1,1]$ we have that*

$$|T_\Gamma(f_0, f_1, \dots, f_m, 1, \dots, 1)(\lambda)| \leq ||f_m||_{U^1(L)} + O_\Gamma(\varepsilon),$$

where $1 = 1_{[0,1]^d}$.

Let $f_A = 1_A - \alpha 1_{[0,1]^d}$.

Corollary 1 *Let Γ be a proper k-degenerate distance graph with $n+1$ vertices in $\mathbb{R}^d$ with $d \geq k+1$. Let $\alpha \in (0,1)$ and $0 < \lambda \leq \varepsilon \ll_\Gamma \alpha^{n+1}$. If $A \subset [0,1]^d$ with $|A| = \alpha$ satisfies $||f_A||_{U^1(\varepsilon^6\lambda)} \ll \varepsilon$, then*

$$T_\Gamma(1_A, 1_A, \dots, 1_A)(\lambda) \geq \frac{c_0}{2}\alpha^{n+1}$$

where

$$c_0 = \iint \cdots \int d\mu_n(x_n) \cdots d\mu_1(x_1) dx.$$