Random triangles and polygons in the plane

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February 23, 2017

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Dictionary

Measures.

- **(Outer) measure** μ on space X is a set function $\mu\colon 2^X \to [0, \infty]$ such that
  1. $\mu(\emptyset) = 0$
  2. $\mu(A) \leq \mu(B)$ whenever $A \subseteq B$
  3. $\mu(\bigcup_{i=1}^{\infty} A_n) \leq \sum_{i=1}^{\infty} \mu(A_n)$
- Measure space (X, μ) (or just measure μ) is **invariant** under $\mathcal{F}$, where $\mathcal{F}$ is some class of transformations $X \to X$, if for every $A \subseteq X$ and every $f \in \mathcal{F}$ we have that $\mu(A) = \mu(f(A))$.
- Measure μ on metric space (X, d) is **uniformly distributed** if for every $x, y \in X$ and every $r > 0$ it holds that $\mu(B(x, r)) = \mu(B(y, r))$, where $B(x, r)$ denotes the ball of radius r centred at x.
- Let (X, μ) be a measure space and $f\colon X \to Y$ a mapping. Then f defines a **pushforward measure** on Y, denoted by $f_\sharp\mu$, by relation $f_\sharp\mu(A) = \mu(f^{-1}(A))$ for every $A \subseteq Y$.
- **Fact:**(Haar's theorem) Let G be a compact topological group. Then there is a *unique* (Radon) probability measure μ on G that is invariant under the operation of G.

Group actions. Let G be a group acting on X. Then

- the **orbit** of $x \in X$ is the set $Orbit_x := \{g(x)\colon g \in G\}$.
- the **stabilizer** of $x \in X$ is the *subgroup* $Stab_x := \{g \in G\colon g(x) = x\}$.
- the **stabilizer** of $A \subseteq X$ is the *subgroup* $Stab_A := \{g \in G\colon g(A) \subseteq A\}$.
- **Fact:** (Orbit-stabilizer theorem) If X is finite, then $|Orbit_x| = |G| / |Stab_x|$.

We say that the action of G on a metric space (X, d) is

- **transitive** if for every $x, y \in X$ there is $g \in G$ such that $g(x) = y$. In other words, there is only one orbit in X.
- **distance-preserving** if for every $x, y \in X$ and every $g \in G$ it is true that $d(x, y) = d(g(x), g(y))$. In other words, all elements of G define an isometry of X.

Important groups:

- **Orthogonal group** $O(n)$ consists of all linear isometries of $\mathbb{R}^n$. That is, consists of all rotations and reflexions along hyperplanes going through the origin. In the language of matrices, it consists of all orthogonal matrices.
- **Hyperoctahedral group** $B(n)$ is a finite subgroup of $O(n)$ consisting of signed permutation matrices. In other words, it consists of matrices that are of the form $\mathrm{diag}(\pm 1, \pm 1, \ldots, \pm 1) \times P$, where P is a permutation matrix.

Grassmannian manifold.

- Real projective space $\mathbb{R}P^n$ is a space of lines through the origin in $\mathbb{R}^{n+1}$ with the linear structure inherited from $\mathbb{R}^{n+1}$.
- The **Grassmannian manifold** $G_r(\mathbb{R}^n)$ is the space of r-dimensional linear subspaces of $\mathbb{R}^n$.
  - $G_1(\mathbb{R}^n) \cong G_{n-1}(\mathbb{R}^n) \cong \mathbb{R}P^{n-1}$
  - It is a *compact metric space* with the metric $d(U, V) := \|P_U - P_V\|$, where $U, V \in G_r(\mathbb{R}^n)$, P_U is the orthogonal projection onto U and $\|\cdot\|$ is the operator norm arising from the Euclidean norm.
  - The action of $O(n)$ on $G_r(\mathbb{R}^n)$ is *transitive* and *distance-preserving.*
- Let $P \in G_2(\mathbb{R}^n)$ with orthonormal basis $(\vec{u}, \vec{v})$. We define **Plücker coordinates** for P as a skew-symmetric matrix $\Delta(P)$ such that $\Delta(P)_{i,j} := \begin{vmatrix} u_i & v_i \\ u_j & v_j \end{vmatrix}$. This is well-defined up to scalar multiplication.
- We call $P \in G_2(\mathbb{R}^n)$ a **semi-circular lift** of a polygon in $\mathbb{R}^2$ if there is orthonormal basis $(\vec{u}, \vec{v})$ of P such that all vectors (u_i, v_i) lie inside the semi-circle starting at (u_1, v_1) and going counter-clockwise to $(-u_1, -v_1)$.

Note: The pictures here are taken, together with their description from the article.

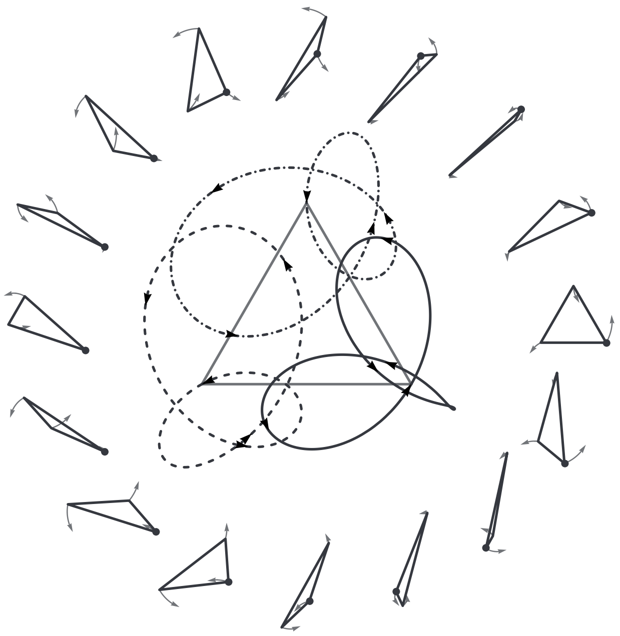

Figure 1: These are two different visualizations of the triangle motion induced by rotating the point $\frac{1}{\sqrt{3}}(1,1,1)$ corresponding to the equilateral triangle around the axis $\left(-1,1,-\sqrt{2}\right)$. The circle of triangles shows 16 equally-spaced points along the resulting great circle together with the path each vertex will traverse in the next time step. The figure in the middle shows the starting equilateral triangle along with the three curves traced out by the vertices. The solid curve is the path of the vertex marked with a dot in the outside triangles.

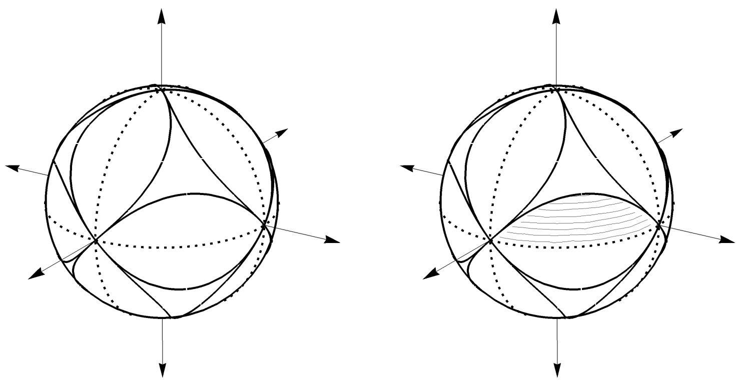

Figure 2: The right triangles are the heavy black curves on the sphere (the dotted lines indicate the intersections of the sphere with the coordinate planes). The hatched region in the right hand figure shows $\frac{1}{24}$ of the region of obtuse triangles. We compute the area of this region below.