Variations on the sum-product problem

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Introduction, main results

Let $A, B, C \subset \mathbb{R}$. We define the *sum set* $A + B = \{a + b : a \in A, b \in B\}$ and the *product set* $AB = \{ab : a \in A, b \in B\}$. We focus our study on the set $A(B + C) = \{a(b + c) : a \in A, b \in B, c \in C\}$, particularly we are interested in the set $A(A + A)$, where both additive and multiplicative structure is present. While the best lower bound $\max\{|AA|, |A+A|\} \gg \frac{|A|^{4/3}}{(\log |A|)^{1/3}}$ implies

$$|A(A + A)| \gg \frac{|A|^{4/3}}{(\log |A|)^{1/3}},$$

Balog conjectures that $|A(A + A)| \gg |A|^{2-\varepsilon}$ for any $\varepsilon > 0$. We prove

Theorem 1

$$|A(A + A)| \gtrapprox |A|^{\frac{3}{2}+\frac{1}{178}}.$$

The notation $\gtrapprox$ is used to supress both constant and logarithmic factors. Our method can be addapted for the bounds where more variables are involved. We obtain two results:

Theorem 2

$$|A(A + A + A)| \gtrapprox |A|^{\frac{7}{4}+\frac{1}{284}}$$

Theorem 3

$$|A(A + A + A + A)| \gg \frac{|A|^2}{\log |A|}.$$

Especially the last bound is important since it verifies extended version of Balog's conjecture. Moreover, this bound is tight in the case when A is an arithmetic progression.

Notation and auxiliary lemma

Given $x \in \mathbb{R}$, we use the notation $r_{A+B}(x)$ to denote the number of representations of x as an element of $A + B$, more precisely

$$r_{A+B}(x) = |\{(a, b) \in A \times B : a + b = x\}|.$$

In a similar way we define the number of representations of x as an element of a given set in the subscript, for example $r_{A(B+C)}(x)$.

We denote the *multiplicative energy* $E^*(A)$ of A as the number of solutions to the equation $a_1a_2 = a_3a_4$, such that $a_1, a_2, a_3, a_4 \in A$. Similarly, the *multiplicative energy* $E^*(A, B)$ of A and B is the number of solutions to the equation $a_1b_1 = a_2b_2$, such that $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

There is a connection between multiplicative energy and multiplicative representation function since

$$E^*(A, B) = \sum_x r^2_{AB}(x) = \sum_x r_{A:A}(x)r_{B:B}(x) = \sum_x r^2_{A:B}(x).$$

Lemma 1 If $|X| \le |A||B|$ then

$$\sum_{x\in X} E^+(A, xB) \ll |A|^{3/2}|B|^{3/2}|X|^{1/2}$$

Lemma 2 If $|A|, |B| \geq 4$ then

$$E_2^*(A)|A(B+C)|^2 \gg \frac{|A|^4|B||C|}{\log |A|}.$$

Prerequisites

Proposition 1 (Szemerédi-Trotter)

$$I(P, L) \ll |P|^{2/3}|L|^{2/3} + |L| + |P|$$

Any point of P incident to at least t lines we call t-rich point. Let P_t denote the set of all t-rich points.

Corollary 1 We have

$$|P_t| \ll \frac{|L|^2}{t^3} + \frac{|L|}{t}.$$

Moreover, if no point of P_t is incident to more than $|L|^{1/2}$ lines, then

$$|P_t| \ll \frac{|L|^2}{t^3}.$$

Proposition 2

$$|A:A|^{10}|A+A|^9 \gtrapprox |A|^{24} \quad \text{and} \quad |A:A|^6|A-A|^5 \gtrapprox |A|^{14}$$

Proposition 3 If $E^*(A) \geq |A|^3/K$ then there is $A' \subset A$ such that $|A'| \gtrapprox |A|/2$ and

$$|A':A'| \lessapprox K^4 \frac{|A'|^3}{|A|^2}.$$

Proposition 4

$$E^*(A) \ll |A+A|^2 \log |A|.$$