Short Proofs of Some Extremal Results

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Induced forests in sparse bipartite graphs

Theorem 1 Let d be a positive integer. Every bipartite graph G on n vertices with average degree at most d contains an induced subgraph on at least $(1/2+\delta)n$ vertices with $\delta = (2^7 d^2)^{-4d}$.

Ramsey saturated graphs

Ramsey number $r(G)$ of a graph G is the smallest natural number N such that every two-coloring of the edges of the complete graph K_N contains a monchromatic copy of G. Graph G on n vertices is *Ramsey unsaturated* if there exist an edge $e \in E(K_n) \setminus E(G)$ such that $r(G+e) = r(G)$. If $r(G+e) > r(G)$ for all edges $e \in E(K_n) \setminus E(G)$, we say that G is *Ramsey saturated.*

Theorem 2 There exist $c > 0$ such that there are at least 2^{cn^2} non-isomorphic Ramsey saturated graphs on n vertices.

Lemma 1 For any graph G with n vertices and m edges we have $r(G) > 2^{m/n-1}$.

Lemma 2 For any bipartite graph G with n vertices and maximum degree Δ we have $r(G) < \Delta 2^{\Delta+5} n$.

Degeneracy and online Ramsey theory

Online Ramsey game is a game of Painter and Builder on a board with infinite independent set of vertices. At each step Builder exposes an edge and Painter decides whather to color it red or blue. Builder's aim is to force Painter to draw a monochromatic copy of a fixed graph G. We consider q-color online Ramsey game.

A graph G is *d-degenerate*, if every subgraph of it has a vertex of degree at most d. Equivalently, there is an ordering of vertices od G, say $u_1, u_2, \ldots, u_n$, such that for each $1 \le i \le n$ the vertex u_i has at most d neighbours u_j with $j < i$.

Theorem 3 In the q-color online Ramsey game, Builder may force Painter to draw a monochromatic copy of any d-degenerate graph while only drawing a $(qd-(q-1))$-degenerate graph.

Clique partitions of very dense graphs

A *clique partition* of graph G is a collection of complete subgraphs of G that partition the edge set of G. The *clique partition number* $\mathrm{cp}(G)$ is the smallest number of cliques in a clique partition of G. Let $\bar{G}$ denotes a complement of graph G.

Theorem 4 If F is a forest on n vertices then $\mathrm{cp}(\bar{F}) = O(n \log\log n)$.

Steiner (n,k)-system is a family of k-element subsets of an n-element set such that each pair apears in exactly one of the subsets. One can view Steiner (n,k)-system as a partition of K_n into cliques of size k.

Theorem 5 If F has n vertices and $m \ge \sqrt{n}$ edges then $\mathrm{cp}(\bar{F}) = O((mn)^{2/3})$.

Corollary 1 If F has n vertices and $o(n^2)$ edges then $\mathrm{cp}(\bar{F}) = o(n^2)$.

Lemma 3 For s, t, n with $s+t \leq n$ we have $\mathrm{cp}(K_n \setminus (K_s \cup K_t)) \geq st$.

Lemma 4 If G has n vertices and all but $t \geq 1$ vertices of G have degree $n-1$ then $\mathrm{cp}(G) < tn$.

Lemma 5 Let $f(n,k)$ denote the minimum number of cliques each on at most k vertices needed to clique partition K_n. If $n > k$ then $f(n,k) = \Theta(\max((n/k)^2, 2))$.

A *tree partition* of a graph G is a collection $\{T_1, \dots, T_r\}$ of subtrees of G such that each edge of G is in exactly one tree and T_i shares with T_j, $i < j$, at most one vertex.

Lemma 6 Let T be a tree on n vertices and $2 \leq k \leq n$. Then there exist a tree partition $\{T_1, \dots, T_r\}$ of T into at most $2n/k$ trees such that the number of vertices of each T_i is between $k/3$ and k.

Hilbert cubes in dense sets

A Hilbert cube is a set $H \subset \mathbb{N}$ of the form

$$H = H(x_0, x_1, \dots, x_d) = \{x_0 + \sum_{i \in I} x_i : I \subset [d]\}.$$

We refer to d as the *dimension of* H. The smallest n such that any coloring of the set $[n]$ with a fixed number r colors must contain a monochromatic Hilbert cube of dimension d we call $h(d,r)$.

Theorem 6 For any $0 < \delta < 1$ there exists $c > 0$ such that with high probability a random set of $[n]$, where each element is chosen independently with probability δ, does not contain Hilbert cubes of dimension $c\sqrt{\log n}$.

Let X be a set with elements $1 \leq x_1 < x_2 < \cdots < x_d$. We write

$$\sum(X) = \{\sum_{i \in I} x_i : I \subset [d]\}.$$

Lemma 7 For any set X with d elements we have

$$\binom{d+1}{2} + 1 \leq |\sum(X)| \leq 2^d.$$

A *generalized arithmetic progression* (GAP) is a subset Q of $\mathbb{Z}$ of the form $Q = \{x_0 + a_1 x_1 + \cdots + a_r x_r : 1 \leq a_i \leq A_i\}$. We refer to r as the *rank of* Q and the product $A_1 \cdot \dots \cdot A_r$ as the *volume of* Q.

Lemma 8 For every $C > 0$ and $0 < \varepsilon < 1$ there exist positive constants r and C' such that if X is a multiset with d elements and $|\sum(X)| \leq d^C$ then there is a GAP Q of dimension r and volume at most $d^{C'}$ such that all but most $d^{1-\varepsilon}$ elements of X are in Q.

Lemma 9 For $s \leq \log d$ the number of d-sets $X \subset [n]$ with $|\sum(X)| \leq 2^s d^2$ is at most $n^{O(s)} d^{O(d)}$.