Improved Cheeger's Inequality: Analysis of Spectral Partitioning Algorithms through Higher Order Spectral Gap

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Cheeger's Inequality

- **conductance** of a subset S od vertices of a d-regular graph is given by

$$\Phi(S) := \frac{|E(S, \bar{S})|}{\mathrm{d}min\{|S|, |\bar{S}|\}}$$

- **conductance of a graph G** is

$$\Phi(G) := min_{S \in V} \Phi(S)$$

- **Cheeger's inequality** says: for the normalized Laplacian matrix $L := I - \frac{1}{d}A$ and its eigenvalues $0 = \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_n \leq 2$ holds:

$$\frac{1}{2}\lambda_2 \leq \Phi(G) \leq \sqrt{2\lambda_2}$$

Improved Cheeger's Inequality

$$\Phi(G) = O(k)\frac{\lambda_2}{\sqrt{\lambda_k}}$$

Definitions & Notation Used

- **weight** function $w : E \to (0, \infty)$
  - $F \subseteq E$: $w(F) := \sum_{e \in F} w(e)$
  - $v \in V$: $w(v) := \sum_{u \sim v} w(u, v)$
- **volume** of $S \subseteq V$: $vol(S) := \sum_{v \in S} w(v)$
- **Dirichlet conductance** of $S \subseteq V$ is:

$$\Phi(S) := \frac{w(E(S, \bar{S}))}{min\{vol(S, \bar{S})\}}$$

- **threshold set** for $f \in \mathbb{R}^n$ is $t \in \mathbb{R}$, $V_f(t) := \{v : f(v) \geq t\}$
- **volume** on an interval I is $vol_f(I) := vol(V_f(t))$
- **Dirichlet conductance** of funciton f is:

$$\Phi(f) := min_{t \in \mathbb{R}} \Phi(V_f(t))$$

- for $t_1, t_2, \ldots t_l \in \mathbb{R}$, let $\psi : \mathbb{R} \to \mathbb{R}$ be:

$$\psi_{t_1, t_2, \ldots t_l}(x) := argmin_{t_i}|x - t_i|$$

- $\rho-$ **Lipschitz** with respect to f is a function g if

$$|g(u) - g(v)| \leq \rho|f(u) - f(v)|$$

Improved Cheeger's Inequality we Prove

$$\Phi(f) = O(k)\frac{\lambda_2}{\sqrt{\lambda_k}}$$

Some Spectral Theory

- $l^2(V,w)$ is the Hilbert space of functions $f: V \to \mathbb{R}$ with inner product $< f, g >_w := \sum_{v\in V} w(v)f(v)g(v)$, norm $||f||_w^2 :=< f, f >_w$
- **adjacency operator**: $Af(v) := \sum_{u\sim v} w(u,v)f(u)$
- **diagonal operator**: $Df(v) := w(v)f(v)$
- **combinatorian Laplacian**: $L := D - A$, **normalized Laplacian**: $L_G := I - D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$
- **Rayleigh quotient**: let f and g be functions fulfilling $g = D^{-\frac{1}{2}}f$, then

$$R_G(f) := \frac{\sum_{u\sim v} w(u,v)|f(u)-f(v)|^2}{||f||_w^2} = \frac{< g, L_G g >}{< g, g >}$$

- **eigenvalues of** L_G: $\lambda_k = min_{f_1,\dots f_k \in l^2(V,w)} max_{f\neq 0}\{R(f) : f \in span\{f_1,\dots f_k\}\}$, where $f_1,\dots f_k$ are independent

Lemmas, Claims & Notation used for the Proof

Lemma. 2.1 There $\exists$ two disjointly supported functions $f_+, f_- \in l^2(V,w)$ such that $f_+ \geq 0$ and $f_- \leq 0$ and $R(f_+) \leq \lambda_2$ and $R(f_-) \leq \lambda_2$.

Corollary. 2.2 There $\exists$ a function $f \in l^2(V,w)$ such that $f \geq 0, R(f) \leq \lambda_2, supp(f) \leq \frac{vol(V)}{2}$ and $||f||_w = 1$.

Lemma. 2.3 For any k disjointly supported functions $f_1, f_2, \dots f_k \in l^2(V,w)$ we have:

$$\lambda_k \leq 2max_{1\leq i\leq k} R(f_i).$$

Lemma. 2.4 For $\forall$ non-negative $h \in l^2(V,w)$ such that $supp(h) \leq \frac{vol(V)}{2}$ holds:

$$\Phi(h) \leq \frac{\sum_{u\sim v} w(u,v)|h(u)-h(v)|}{\sum_v h(v)w(v)}.$$

Lemma. 3.1 There $\exists$ a $2k+1$-step approximation of f, call it g, such that

$$||f-g||_w^2 \leq \frac{4R(f)}{\lambda_k}.$$

Lemma. 3.2 For $\forall$ $2k+1$-step approximation of f with $||f||_w = 1$, call it g, holds:

$$\Phi(f) \leq 4kR(f) + 4\sqrt{2}k||f-g||_w\sqrt{R(f)}.$$

- $\mu : \mathbb{R} \to \mathbb{R}$, $\mu(x) := |x - \psi_{t_0,\dots t_{2k}}(x)|$
- $$h(v) := \int_0^{f(v)} \mu(x)dx.$$

Claim. 3.3 For $\forall v \in V$:

$$h(v) \geq \frac{1}{8}f^2(v).$$

Claim. 3.4 For $\forall u, v \in V$:

$$|h(v)-h(u)| \leq \frac{1}{2}|f(v)-f(u)|(|f(u)-g(u)| + |f(v)-g(v)| + |f(v)-f(u)|).$$