Tomáš Masařík
tarken@kam.mff.cuni.cz

Presented paper by Noga Alon

Bipartite decomposition of random graphs

(http://arxiv.org/abs/1402.6466)

Definitions

Definition (Maximal size complete bipartite induced subgraph).
$\beta(G) :=$ size of maximal complete bipartite induced subgraph of G.

Definition (Minimal bipartite decomposion number).
$\tau(G) :=$ minimal number of complete edge disjoint covering bipartite subgraphs of G.

Definition (Minimal nontrivial bipartite decomposion number).
$\tau'(G) :=$ minimal number of complete edge disjoint covering nontrivial (non-star) bipartite subgraphs of G. (∞ if didn't exist)

Definition .
$\gamma(G) := |V| -$number of C_4 forming G.

Main theorems

Conjecture *(Erdös disproved conjecture)*
For $G \in G(n, \frac{1}{2})$ whp

$$\tau(G) = n - \alpha(G).$$

Theorem *(1).*

$$f(k) := \binom{n}{k} 2^{-\binom{k}{2}},$$

$$k_0 := \max k \ st. \ f(k) \geq 1.$$

For $G \in G(n, \frac{1}{2})$

(i) *If $f(k_0 + 1) << 1 << f(k_0)$ then $\alpha(G) = k_0$ and $\beta(G) = k_0 + 2$ whp.*

(ii) *If $f(k_0) \in \Theta(1)$ then whp one of the folowing, each of them with probability bounded away from 0 and 1.*

  (a) $\alpha(G) = k_0$ *and* $\beta(G) = k_0 + 2$

  (b) $\alpha(G) = k_0$ *and* $\beta(G) = k_0 + 1$

  (c) $\alpha(G) = k_0 - 1$ *and* $\beta(G) = k_0 + 2$

  (d) $\alpha(G) = k_0 - 1$ *and* $\beta(G) = k_0 + 1$

(iii) *If $f(k_0+1) \in \Theta(1)$ then whp one of the folowing, each of them with probability bounded away from 0 and 1.*
Same as previous, set there $k_0 := k_0 + 1$

Theorem *(2).*
$\exists c > 0$ st. $\forall p$ st. $\frac{2}{n} \leq p \leq c$ then $G \in G(n, p)$ whp satisfies

$$\tau(G) = n - \Theta(\frac{\log{(np)}}{p}).$$

Theorem *(3).*
If $p << n^{-\frac{7}{8}}$ *then* $G \in G(n,p)$ *whp satisfies*

$$\tau(G) = n - \max_{G \supseteq H induced,\ st.\ components\ are\ C_4\ or\ v \in V} (\gamma(H)).$$

Some lemmas

Theorem *(Stein-Chen method).*
Let $\{X_\alpha\}_{\alpha\in\mathcal{F}}$ *be a finite family of indicator vatiables with dependency graph* L *and* $X := \sum_{\alpha\in\mathcal{F}} X_\alpha$ *having* $E[X] = \sum_{\alpha\in\mathcal{F}} E[X_\alpha] = \lambda$. *Let* $PO(X)$ *be a Poisson random variable with expectation* λ, *then:*

$$\sup_{\alpha\in\mathcal{F}} |P(X_\alpha) - PO(X_\alpha)| \leq \min(\lambda^{-1}, 1)(\sum_{\alpha\in\mathcal{F}} E[X_\alpha]^2 + \sum_{\alpha,\beta\in\mathcal{F},(\alpha,\beta)\in E(L)} (E[X_\alpha X_\beta] + E[X_\alpha]E[X_\beta])).$$

Lemma *(Trivial vs. non-trivial).*
$\forall G(v,E) \exists U \subseteq V$ *st.*

$$\tau(G) = |V| - |U| + \tau'(G).$$

Theorem *(Čebyšev inequality).*

$$P(|X - E[X]| \geq E[X]) \leq \frac{Var[X]}{(E[X])^2}$$

Lemma *(About variance).*

$$Var[X] = E[X^2] - (E[X])^2,$$
$$Cov[X,Y] = E[XY] - E[X]E[Y],$$
$$Var[\sum_i X_i] = \sum_i Var[X_i] + \sum_{i\neq j} Cov[X_i, X_j].$$

Lemma *(Technical lemma).*
$\exists a, c, C;\ \forall p$ *st.* $0 \leq p \leq c$ *and* $np \geq C \log n$ *then*

$$P(\tau'(G) \leq 2n) \leq 2^{-apn^2}.$$

Lemma *(relation of expectation of IS and BG).*
Define

$$g(k) := \binom{n}{k+2}(2^{k+1} - 1)2^{-\binom{k+2}{2}}.$$

Then $g(k) \in \Theta(f(k))$.

Useful estimates

Lemma

$$1 \geq \binom{n}{k_0+1} 2^{-\binom{k_0+1}{2}} \geq (\frac{n}{k_0+1})^{k_0+1} 2^{-\binom{k_0+1}{2}}.$$
$$1 \leq \binom{n}{k_0} 2^{-\binom{k_0}{2}} \leq (\frac{ne}{k_0})^{k_0} 2^{-\binom{k_0}{2}}.$$
$$n = \Theta(k_0 2^{\frac{k_0}{2}}).$$