AN APPROXIMATE VERSION OF SIDORENKO'S CONJECTURE

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Definitions:

- a (small) bipartite graph H, a (large) graph G
- *homomorphism density*: $t_H(G) =$ (number of homomorphism $H \to G)/|G|^{|H|}$
- *subgraph density*: H*-density* of $G =$ a fraction of injective mappings $H \to G$ that are homomorphisms

Observation. For dense G, the H-density of $G = t_H(G) + o(1)$.

Conjecture [Sidorenko, Erdős-Simonovits]. Let H be a bipartite graph with m edges. For every graph G,

$$t_H(G) \geq t_{K_2}(G)^m.$$

That is, among the graphs G of edge density p, $t_H(G)$ attains its minimum when G is a random graph of edge density p.

Conjecture (analytic form). Let μ be the Lebesgue measure on $[0, 1]$ and let $h(x, y)$ be a bounded, non-negative, symmetric and measurable function on $[0, 1]^2$. Let H be a bipartite graph with vertices $u_1, \ldots, u_t$ in the first part, vertices $v_1, \ldots, v_s$ in the second part and m edges. Then

$$\int \prod_{(u_i,v_j)\in E(H)} h(x_i, y_j) d\mu^{s+t} \geq \left(\int h d\mu^2\right)^m.$$

known: e.g. for complete bipartite graphs, trees, even cycles, subgraphs of $K_{3,s}$ (and perhaps $K_{4,s}$) [Sidorenko, 1993], hypercubes [Hatami, 2010]

Theorem 1. Sidorenko's conjecture holds for every bipartite graph H which has a vertex complete to the other part.

Corollary (approximate Sidorenko's conjecture). If H is a bipartite graph with m edges and width w (minimum degree of the bipartite complement $\overline{H}$), then $t_H(G) \geq t_{K_2}(G)^{m+w}$ holds for every graph G.

Definitions: A sequence $(G_n : n = 1, 2, \ldots)$ of graphs is called *quasirandom* with density p (where $0 < p < 1$) if, for every graph H,

$$t_H(G_n) = (1 + o(1))p^{|E(H)|}. \tag{1}$$

A graph F is called *forcing* if the fact that (1) holds for $H = K_2$ and $H = F$ implies that $(G_n : n = 1, 2, \ldots)$ is quasirandom.

known: C_{2t} and $K_{2,t}$ are forcing [Chung, Graham, Wilson, 1989];
$K_{s,t}$ are forcing [Skokan, Thoma, 2004]

Conjecture (forcing). A graph is forcing if and only if it is bipartite and contains a cycle.

Theorem 2. The forcing conjecture holds for every bipartite graph H which has two vertices in one part complete to the other part, which has at least two vertices.

Proof of Theorem 1.

Lemma 1. (dependent random choice) Let G be a graph with N vertices and $pN^2/2$ edges. Call a vertex v bad with respect to k if the number of sequences of k vertices in $N(v)$ with at most $(2n)^{-n-1}p^kN$ common neighbors is at least $\frac{1}{2n}|N(v)|^k$. Call v good if it is not bad with respect to k for all $1 \leq k \leq n$. Then the sum of the degrees of the good vertices is at least $pN^2/2$.

Lemma 2. Suppose $\mathcal{H}$ is a hypergraph with v vertices and at most e edges and $\mathcal{G}$ is a hypergraph on N vertices with the property that for each k, $1 \leq k \leq v$, the number of sequences of k vertices of $\mathcal{G}$ that do not form an edge of $\mathcal{G}$ is at most $\frac{1}{2e}N^k$. Then the number of homomorphisms from $\mathcal{H}$ to $\mathcal{G}$ is at least $\frac{1}{2}N^v$.

Lemma 3. Let $H = (V_1, V_2, E)$ be a bipartite graph with n vertices and m edges such that there is a vertex $u \in V_1$ which is adjacent to all vertices in V_2. Let G be a graph with N vertices and $pN^2/2$ edges, so $t_{K_2}(G) = p$. Then the number of homomorphisms from H to G is at least $(2n)^{-n^2}p^mN^n$.

last step: "tensor power trick" to eliminate the constant $(2n)^{-n^2}$

Observation. For all H, G, F, $t_H(F \times G) = t_H(F) \times t_H(G)$.

On the logarithmic calculus and Sidorenko's conjecture

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Theorem 3. Let H be the (bipartite) graph on the vertex set $\{x, y_1, y_2, \ldots, y_m, v_1, v_2, \ldots, v_k\}$ such that x is connected to $v_1, v_2, \ldots, v_k$ and y_t is connected to the vertices $S_t \subseteq \{v_1, v_2, \ldots, v_k\}$ where $|S_t| = a_t$. Let $e = k + \sum_{t=1}^{m} a_t$ be the total number of edges in H. If $W : [0,1]^2 \to \mathbb{R}^+$ is a measurable function and $p = \mathbb{E}(W)$, then

$$t(H, W) := \mathbb{E} \prod_{(x_i, x_j) \in E(H)} W(x_i, x_j) \geq p^e.$$

Theorem 4. The forcing conjecture holds for bipartite graphs in which one vertex is complete to the other side (and are not trees).

Main tool: Jensen's inequality. Let (Ω, μ) be a probability space, let c be a convex (resp. concave) function on an interval $D \subset \mathbb{R}$ and $g : \Omega \to D$ be a measurable function. Then

$$\mathbb{E}(c(g)) \geq c(\mathbb{E}(g)) \quad \text{(convex)}, \quad \mathbb{E}(c(g)) \leq c(\mathbb{E}(g)) \quad \text{(concave)}. \tag{2}$$

Moreover if $\mathbb{E}(f) = 1$ for some non-negative function f on Ω then also

$$\mathbb{E}(fc(g)) \geq c(\mathbb{E}(fg)) \quad \text{(convex)}, \quad \mathbb{E}(fc(g)) \leq c(\mathbb{E}(fg)) \quad \text{(concave)}. \tag{3}$$

If c is a strictly convex (concave) function then equality in (3) is only possible if g is constant on the support of f.