ON SUNFLOWERS AND MATRIX MULTIPLICATION

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Definition (k-sunflower)**.** Subsets $A_1, \dots, A_k$ of a universe U form a k-sunflower if $\forall i \neq j$ $A_i \cap A_j = \cap_{i=1}^k A_i$.

Conjecture 1 (Classical sunflower conjecture)**.** *For every $k > 0$ there is a constant c_k such that the following holds. Let $\mathcal{F}$ be an arbitrary family of sets of size s from some universe U. If $|\mathcal{F}| \geq c_k^s$ then $\mathcal{F}$ contains a k-sunflower.*

Conjecture 2 (Sunflower conjecture in $\{0,1\}^n$)**.** *There is $\epsilon > 0$ such that any family $\mathcal{F}$ of subsets of $[n]$ of size $|\mathcal{F}| \geq 2^{(1-\epsilon)n}$ contains a 3-sunflower.*

Definition (Sunflowers in $\mathbb{Z}_D^n$)**.** We say that k vectors $v_1, \dots, v_k \in \mathbb{Z}_D^n$ form a k-sunflower if for every coordinate $i \in [n]$ it holds that either $(v_1)_i = \cdots = (v_k)_i$ or they all differ on that coordinate.

Conjecture 3 (Sunflower conjecture in $\mathbb{Z}_D^n$)**.** *For every k there is an absolute constant b_k so that for every D and every n, any set of at least b_k^n vectors in $\mathbb{Z}_D^n$ contains a k-sunflower.*

Conjecture 4 (Weak sunflower conjecture in $\mathbb{Z}_D^n$)**.** *There is an $\epsilon > 0$ so that for $D > D_0$ and $n > n_0$, any set of at least $D^{(1-\epsilon)n}$ vectors in $\mathbb{Z}_D^n$ contains a 3-sunflower.*

Conjecture 5 (Weak sunflower conjecture in $\mathbb{Z}_3^n$)**.** *There is an $\epsilon > 0$ so that for $n > n_0$, any set of at least $3^{(1-\epsilon)n}$ vectors in $\mathbb{Z}_3^n$ contains a 3-sunflower.*

Definition (Multicolored sunflower)**.** Triple $(x, y, z) \in \mathbb{Z}_3^n \times \mathbb{Z}_3^n \times \mathbb{Z}_3^n$ is an ordered sunflower in $\mathbb{Z}_3^n \times \mathbb{Z}_3^n \times \mathbb{Z}_3^n$ if $\{x, y, z\}$ is a sunflower in $\mathbb{Z}_3^n$. We say that ordered sunflowers $(a, b, c), (x, y, z)$ are disjoint, if $a \neq x, b \neq y$ and $c \neq z$. We say that a collection of ordered triples contains a multicolored sunflower if it contains three triples $(x^{(1)}, y^{(1)}, z^{(1)}), (x^{(2)}, y^{(2)}, z^{(2)}), (x^{(3)}, y^{(3)}, z^{(3)})$ where $\{x^{(1)}, y^{(2)}, z^{(3)}\}$ is a sunflower.

Conjecture 6 (Multicolored sunflower conjecture in $\mathbb{Z}_3^n$)**.** *There is an $\epsilon > 0$ so that for $n > n_0$, every collection $\mathcal{F} \subseteq \mathbb{Z}_3^n \times \mathbb{Z}_3^n \times \mathbb{Z}_3^n$ of at least $3^{(1-\epsilon)n}$ ordered sunflower contains a multicolored sunflower.*

Definition (Uniquely solvable puzzle)**.** A uniquely solvable puzzle (USP) of width n is a subset $\mathcal{F} \subseteq \mathbb{Z}_3^n$ satisfying the following property: For all permutations $\pi_{0,1,2} \in Sym(\mathcal{F})$, either $\pi_0 = \pi_1 = \pi_2$ or else there is $u \in \mathcal{F}$ and $i \in [n]$ such that at least two of $(\pi_0(u))_i = 0, (\pi_1(u))_i = 1, (\pi_2(u))_i = 2$ hold.
The USP capacity is the largest constant C such that there exist USPs of size $(C - o(1))^n$ and width n for infinitely many values of n.

Cohn et al. and Coppersmith, Winograd: USP capacity equals $\frac{3}{2^{2/3}}$.

Definition (Strong USP)**.** A strong USP of width n is a subset $\mathcal{F} \subseteq \mathbb{Z}_3^n$ satisfying the following property: For all permutations $\pi_{0,1,2} \in Sym(\mathcal{F})$, either $\pi_0 = \pi_1 = \pi_2$ or else there is $u \in \mathcal{F}$ and $i \in [n]$ such that exactly two of $(\pi_0(u))_i = 0, (\pi_1(u))_i = 1, (\pi_2(u))_i = 2$ hold.
The strong USP capacity is the largest constant C such that there exist strong USPs of size $(C - o(1))^n$ and width n for infinitely many values of n.

Definition (Local strong USP)**.** A local strong USP of width n is a subset $\mathcal{F} \subseteq \mathbb{Z}_3^n$ if for every $u, v, w \in \mathcal{F}$, not all equal, the sets $\{i \mid u_i = 0\}, \{i \mid v_i = 1\}, \{i \mid w_i = 2\}$ do not form a 3-sunflower.
The local strong USP capacity is the largest constant C such that there exist local strong USPs of size $(C - o(1))^n$ and width n for infinitely many values of n.

Conjecture 7 (Strong USP capacity)**.** *The strong USP capacity (and the local strong USP capacity) equals* $\frac{3}{2^{2/3}}$.

Definition. An Abelian group G (with at least two elements) and a subset S of G satisfy the no three disjoint equivoluminous subsets property if: whenever T_1, T_2, T_3 are three disjoint subsets of S, not all empty, they cannot all have the same sum in G:

$$\sum_{g \in T_1} g \neq \sum_{g \in T_2} g \text{ or } \sum_{g \in T_2} g \neq \sum_{g \in T_3} g$$

Coppersmith and Winograd: If there is a sequence of pairs G, S with the no three disjoint equivoluminous subsets property, such that $\log(|G|)/|S|$ approaches 0, then for every $\epsilon > 0$ there is an $O(n^{2+\epsilon})$ time fast matrix multiplication algorithm.

Theorem (#3.2)**.** *If Conjecture 2 holds with* ϵ_0 *then if* G, S *have no three disjoint equivoluminous subsets property then* $|S| \leq \log(|G|)/\epsilon_0$.