The Littlewood-Offord problem in high dimensions and a conjecture of Frankl and Fűredi

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1 Preliminaries

Let $V = \{v_1, \dots, v_n\}$ be a multiset of n vectors in $\mathbb{R}^d$ with lengths at least 1. The chief object of our investigation is the random sum $X_V = \sum_i \xi_i v_i$, where ξ_i take values 1 and -1 with probabilities 1/2 independently of each other.

The *small ball probability* is the quantity

$$p_d(n, \Delta) = \sup_{V,B} P(X_V \in B),$$

where B ranges over all balls of radius Δ and V over all multisets $\{v_1, \dots, v_n\}$ with lengths at least 1.

In the asymptotic formulae in this paper, Δ and d will play the role of constants and n will be the variable.

Let $s = \lfloor \Delta \rfloor + 1$ ("roundend Δ") and $S(n, m)$ the sum of m largest binomial coefficients in n (e.g., for n odd is $S(n, 2) = \binom{n}{\lceil n/2 \rceil} + \binom{n}{\lfloor n/2 \rfloor}$).

2 Survey of results

Theorem 1 (Erdős's L-O inequality, 1945)**.** $p_1(n, \Delta) = 2^{-n} S(n, s)$.

Theorem 2 (Frankl-Fűredi L-O inequality, 1988)**.** *For any fixed d and Δ, $p_d(n, \Delta) = (1 + o(1))2^{-n} S(n, s)$.*

Theorem 3 (Frankl-Fűredi L-O conjecture, proven here)**.** *Let d, Δ be fixed. If $s - 1 \le \Delta < \sqrt{(s-1)^2 + 1}$ and n is sufficiently large (d, Δ), then*

$$p_d(n, \Delta) = 2^{-n} S(n, s).$$

Remark: the assumption on s is necessary.

3 Tools and auxilliaries

Theorem 4 (Useful estimate, here)**.** *Let V be such that for any hyperplane H in $\mathbb{R}^d$, one has $\operatorname{dist}(v_i, H) \ge 1$ for at least k values of $i = 1, \dots, n$. Then for any unit ball B one has*

$$P(X_V \in B) = O_d(k^{-d/2}).$$

Lemma 1 (Bound by Stirling)**.** $2^{-n}S(n,s) \approx n^{-1/2}$.

Theorem 5 (Esséen's concentration inequality, 1966)**.** *Let X be a random variable taking a finite number of values in $\mathbb{R}^d$, let $R,\ \tau > 0$. Then*

$$\sup_{x_0 \in \mathbb{R}^d} P(|X - x_0| \leq R) = O\left(\frac{R}{\sqrt{d}} + \frac{\sqrt{d}}{\tau}\right)^d \int_{\xi \in \mathbb{R}^d :\ |\xi| < \tau} |E(e(\xi \cdot X))| d\xi.$$

Lemma 2. *Let $w_1, \ldots, w_d \in \mathbb{R}$ be such that*

$$\operatorname{dist}(w_j, \operatorname{Span}\{w_1, \ldots, w_{j-1}\}) \geq 1$$

for each j in $1, \ldots, d$. Then for any $\lambda > 0$,

$$\int_{\zeta \in \mathbb{R}^d :\ |\zeta| < 1} \exp\left(-\lambda \sum_j \|\zeta \cdot w_j\|^2\right) d\zeta = O((1+\lambda)^{-d/2}).$$