New results for the growth of sets of real numbers

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$A \subseteq \mathbb{R}$ finite; $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$; $X \ll Y$–there is an absolute constant C with $X \leq CY$

Definition 1 *An n-variable* expander *is a function f for which the set $f(A) = \{f(a_1, \dots, a_n) \colon a_i \in A\}$ has cardinality at least $\Omega(|A|^{1+\delta})$ for some absolute $\delta > 0$.*

Definition 2 *For distinct $a, b, c, d \in \mathbb{R}$ the* cross ratio *$X(a, b, c, d)$ is defined by*

$$X(a, b, c, d) = \frac{(a-b)(c-d)}{(b-c)(a-d)}.$$

Theorem 3 *Let $f(a, b, c) = X(0, a, b, c) \in \mathbb{R}$. Then $|f(A)| \gg \frac{|A|^2}{\log |A|}$.*

Theorem 4 *Let $g(a, b, c, d) = X(a, b, c, d) \in \mathbb{R}$. Then $|g(A)| \gg |A|^2$.*

Theorem 5 *Let $h(a, b, c, d, e) = (X(a, b, c, d), X(a, b, c, e)) \in \mathbb{R}^2$. Then $|h(A)| \gg \frac{|A|^4}{\log |A|}$.*

Question 6 *Can Theorems 3, 4 and 5 be improved?*

Towards the proofs

Definition 7 *The group $PGL_2(\mathbb{R})$ of* projective transformations *of $\overline{\mathbb{R}}$ is defined by $GL_2(\mathbb{R})/I$, where $I = cI_2, 0 \neq c \in \mathbb{R}$. It has an action on $\overline{\mathbb{R}}$ given by $\left[\begin{pmatrix} p & q \\ r & s \end{pmatrix}\right] x = \frac{px+q}{rx+s}$, which is interpreted in the sense of limits where necessary.*

Lemma 8 *Let $\mathcal{T}$ be the set of ordered triples of distinct elements of $\overline{\mathbb{R}}$. If (a, b, c) and (d, e, f) are both in $\mathcal{T}$ then there is a unique $\tau \in PGL_2(\mathbb{R})$ for which $(\tau(a), \tau(b), \tau(c)) = (d, e, f)$.*

Lemma 9 $X(a_1, a_2, a_3, a_4) = X(b_1, b_2, b_3, b_4)$ *iff $\exists \tau \in PGL_2(\mathbb{R}) \colon \ \tau(a_i) = b_i$ for all i.*

Lemma 10 *(Points lemma) Define $\psi \colon PGL_2(\mathbb{R}) \to \mathbb{PR}^3$ by $\psi\left[\begin{pmatrix} p & q \\ r & s \end{pmatrix}\right] = [p, q, r, s]$. The map ψ is well-defined, injective and its image is $\mathbb{PR}^3 \setminus Q$, where Q is given by $ps = qr$.*

Lemma 11 *(Planes lemma) Let ψ be as in the points lemma. For each $(a, b) \in \mathbb{R}^2$ there is a plane $\pi_{ab} \subseteq \mathbb{PR}^3$ such that if $\tau \in PGL_2(\mathbb{R})$ then $\tau(a) = b$ iff $\psi(\tau) \in \pi_{ab}$. These planes have the following properties:*

1. *Any triple of distinct planes intersects in a single point.*
2. *Each $(a, b) \in \mathbb{R}^2$ determines a unique plane.*
3. *Each pair of distinct planes intersects in a unique line.*
4. *For any $A \subseteq \mathbb{R}$, a point $p \in \mathbb{PR}^3 \setminus Q$ is incident to at most $|A|$ of the planes from $\{\pi_{ab} \colon a, b \in A\}$.*