Dependent Random Choice

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This method is an example of easy application of the celebrated Probabilistic method and could be used in various areas of combinatorics. Roughly speaking, the aim is to find in a given dense graph G a large subset of vertices U such that (almost) all subsets of U of appropriate size have big common neighborhood. Such set U could be used for example for finding an embedding of small sparse graph H in G.

We present some of the variants of using dependent random choice together with some of their applications, mainly in Extremal Combinatorics and Ramsey Theory.

Lemma 1 (Basic variant of Dependent Random Choice)**.** *Let $G = (V, E)$ on n vertices and with average degree $d = 2|E|/n$. If there is a positive integer t such that the formula*

$$\frac{d^t}{n^{t-1}} - \binom{n}{k}\left(\frac{s}{n}\right)^t \geq u$$

holds, then G contains a set $U \subseteq V$ with the following properties:

- *U has size at least u,*
- *every k vertices from U have at least s common neighbors.*

One of the motivations of such a statement could be the following lemma about (greedy) embeddings of bipartite graphs.

Lemma 2. *Let $H = (A \cup B, F)$ be a bipartite graph with $|A| = a$, $|B| = b$, and in which all vertices from B have degree at most k. If a graph G contains a subset of vertices U, $|U| = a$, such that all subsets of U of size k have at least $a + b$ common neighbors, then H is a subgraph of G.*

An easy application of the previous is the following theorem about Turán numbers of bipartite graphs with one part consisting of vertices of bounded degree. By $ex(H, n)$ we will denote the maximum number of edges of an n-vertex graph which does not contain H as a subgraph.

Theorem 3. *If $H = (A \cup B, F)$ is a bipartite graph with all vertices from B of degree at most k, then $ex(n, H) \leq cn^{2-1/k}$, where $c := c(H)$ depends only on H.*

Here are some other applications of the previous tools.

Theorem 4. *If G is a graph with n vertices and ϵn^2 edges, then G contains a 1-subdivision of a complete graph on $\epsilon^{3/2}\sqrt{n}$ vertices.*

Let $r(H)$ be the Ramsey number for H, i.e., it is the minimum number N such that every 2-coloring of edges of K_N contains a monchromatic copy of H.

Theorem 5. *Let Q_r be the r-dimensional hypercube. The Ramsey number $r(Q_r) \leq 2^{3r}$.*

Now let us continue with another version of dependent random choice tuned up for dense graphs which does not contain many triangles.

Lemma 6. *Let $t \geq 2$ be an integer and $G = (V, E)$ be a graph with n vertices and average degree d. If every pair of adjacent vertices of G has at most s common neighbors, then G contains an induced subgraph with at least $d^t/2n^{t-1}$ vertices and average degree at most $2a^t/d^{t-1}$.*

A quick application of this lemma is the following.

Theorem 7. *Every K_5-free graph $G = (V, E)$ on n-vertices contains a triangle-free induced subgraph on $n^{5/12}/2$ vertices.*

It would be nice to have an analogue to Lemma 1 saying, that in a dense graph H we can find U such that both size of U and the number of common neighbors are linear in the size of a graph. Unfortunately, such statement is false. But we can relax a bit the condition that **each** subset of U should have big common neighborhood to prove the following.

Lemma 8 (Dependent Random Choice II)**.** *Let G be a bipartite graph with parts A and B and with $e(G) = c|A||B|$ edges. For any $0 < \epsilon < 1$ there exists a subset $U \subseteq A$ such that*

- *U has size at least $c|A|/2$,*
- *at least a $(1 - \epsilon)$-fraction of the ordered pairs of vertices from U have at least $\epsilon c^2|B|/2$ common neighbors in B.*

Let us note, that this application of the method was used as the main ingredient in the Gowers' proof of Balog-Szemerédi-Gowers theorem.

The last application of dependent random choice which we will present is an extension of the previous one.

Lemma 9 (Dependent Random Choice III)**.** *Fix $\epsilon > 0$ and positive integers d and n, $d \leq n$. If G is a graph with $N > 4d\epsilon^{-d}n$ vertices and at least $\epsilon N^2/2$ edges, then there is a vertex subset U such that*

- *$|U| > 2n$,*
- *the fraction of subsets $S \subset U$ of size d with $|N(S)| < n$ is less than $(2d)^{-d}$.*

We will use this lemma together with the following embedding lemma to obtain an upper bound on Ramsey numbers of bounded degree bipartite graphs.

Lemma 10. *Let H be an n-vertex bipartite graph with maximum degree d. If a graph G contains a subset U with the two properties above, then G contains H as a subgraph.*

This yields the following.

Theorem 11. *Let H be an n-vertex bipartite graph with maximum degree d. If $\epsilon > 0$ and G is a graph with $N \geq c(\epsilon, d)n$ vertices and at least $\epsilon N^2/2$ edges, then H is a subgraph of G, where $c(\epsilon, d) := 8d\epsilon^{-d}$.*

Theorem 12. *If H is an n-vertex bipartite graph with maximum degree d, then $r(H) \leq 8d \cdot 2^d \cdot n$. In particular, the Ramsey numer of the hypercube Q_r is at most $8r \cdot 2^{2r}$, which improves the result of Theorem 5.*