The chance that a convex body is lattice-point free: A relative of Buffon's needle problem

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Definitions.

- $L_{\rho,t}$ ... *integer lattice* $\mathbb{Z}^d$ *rotated by* ρ *and translated by* t
- $\mathcal{L} = \{L_{\rho,t} : \rho \in SO(d), t \in [0,1)^d\}$
- $\mathcal{K}^d$ ... *set of all convex bodies in* $\mathbb{R}^d$
- S^{d-1} ... *$(d-1)$-dimensional unit sphere (boundary of the d-dimensional ball)*
- $w(K, u)$ ... width of K in direction u ... $\max_{x,y \in K} u(x - y)$
- $w(K)$ ... width *of* K ... $\min_{u \in S^{d-1}} w(u)$

Theorem 1. $\forall d \geq 2\ \exists c_1(d), c_2(d) > 0$: *for every* $K \in \mathcal{K}$ *with* $\mathrm{Vol}K \geq c_2(d)$*:*

$$\mathrm{Prob}[K \cap L = \emptyset] \leq \frac{c_1(d)}{\mathrm{Vol}K}.$$

Theorem 2. $\forall d \geq 2\ \exists b_1(d), b_2(d), w_d > 0$: *for every* $K \in \mathcal{K}$ *with* $\mathrm{Vol}K \geq b_2(d)$ *and* $w(K) \leq w_d$*:*

$$\mathrm{Prob}[K \cap L = \emptyset] \geq \frac{b_1(d)}{\mathrm{Vol}K}.$$

Tools.

Definitions.

- octahedron $Oct(a) = \mathrm{conv}\{\pm a_i e_i, \ldots, \pm a_n e_n\} = \{x \in \mathbb{R}^d : \sum_{i=1}^n |x_i/a_i| \leq 1\}$
- slab $S(u, \nu)$, *where* $u \in \mathbb{R}^d, \nu > 0$ ... $\{x : -\nu \leq ux \leq \nu\}$
- $W(K)$ ... lattice width ... $\min_{z \in \mathbb{Z}^d, z \neq 0} w(K, z)$

Theorem 3 (Löwner-John ellipsoid pair)**.** *Given* $K \in \mathcal{K}$*, there exists a pair of ellipsoids* E, E' *such that* $E \subset K \subset E'$*,* E *and* E' *are concentric and* E *is obtained from* E' *by shrinking by a factor of* $1/d$*.*

Corollary 1 (Octahedron pair)**.** *Given* $K \in \mathcal{K}$*, there exists a vector* a *such that for some congruent copy* $K^\star$ *of* K*:*

$$Oct(a) \subset K^\star \subset Oct(d^{3/2}a).$$

Lemma 2. *Let* $A = \{f \in S^{d-1} : \forall i \in [d] : |f_i| \leq \frac{\nu}{a_i|u|}\}$*. We have*

$$\mathrm{Prob}_\rho[Oct(a) \subset \rho S(u, \nu)] = \lambda(A).$$

Let $\alpha_i = \frac{\nu}{a_i|u|}$*. If* $\alpha_i \geq 1$ *for some* i*, then*

$$\prod_{i:\alpha(i)<1} \alpha_i \ll \lambda(A) \ll \prod_{i:\alpha(i)<1} \alpha_i.$$

Theorem 4 (Flatness theorem)**.** *If* $K \in \mathcal{K}^d$ *and* $K \cap \mathbb{Z}^d = \emptyset$*, then* $W(K) \leq W_d$ *(*W_d *depends only on* d*).*

Proof of Theorem 1

- suffices for $K = Oct(a)$ with $a_1 \le a_2/2 \le \cdots \le a_d/2^{d-1}$
- width of $Oct(a)$ is $2\left(\sum_1^d 1/a_i^2\right)^{-1/2} \ge a_1\sqrt{3}$
- $P \subset \mathbb{Z}^d$ ... set of primitive vectors u (g.c.d. of components of u is 1)
- from Flatness theorem:

$$\mathrm{Prob}[Oct(a) \cap L = \emptyset] \le \sum_{u \in \mathbb{Z}^d} \mathrm{Prob}[Oct(a) \subset \rho S(u, W_d/2)]$$

- apply Lemma 2

Proof of Theorem 2

- suffices for $K = Oct(a)$ with $a_1 \le a_2/2 \le \cdots \le a_d/2^{d-1}$
- given $u \in P$, $E(u) := \{\rho \in SO(d) : Oct(a) \subset \rho S(u, 0.48)\}$
- let $P^\star = \{u \in P : 2.1 \le 1/(a_1|u|) \le 2.3\}$ and $P(u) = \{v \in P^\star : |v| \ge |u|, v \ne u\}$

$$\mathrm{Prob}[Oct(a) \cap L = \emptyset] \gg \lambda\left(\bigcup_{u \in p} E(u)\right) \gg \sum_{u \in P^\star}\left(\lambda(E(u)) - \sum_{v \in P(u)} \lambda(E(u) \cap E(v))\right)$$

Applications: randomized integer convex hull

Definitions.
- $u(x)$ ... *volume of the* Macbeath region $K \cap (2x - K)$ *(where $x \in K$)*
- $K(u \le t) = \{x \in K : u(x) \le t\}$
- $E(f_0(I_L(K)))$ ... *expected number of vertices of the randomized integer convex hull*
- $\mathcal{K}_D^d$ ... *set of $K \in \mathcal{K}^d$ with ratio between radii of circumscribed and inscribed circle at most D*

Theorem 5. *Fix $D > 1$. For every $K \in \mathcal{K}_D^d$:*

$$\mathrm{Vol}K(u \le 1) \ll E(f_0(I_L(K))) \ll \mathrm{Vol}K(u \le 1)$$

- It is known that $(\log \mathrm{Vol}K)^{d-1} \ll \mathrm{Vol}K(u \le 1) \ll (\mathrm{Vol}K)^{(d-1)/(d+1)}$ and both bounds can be reached.

Definitions.
- *expected missed area* ... $M(K) = E[\mathrm{Vol}(K \setminus I_L(K))]$
- minimal cap $C(x) = K \cap H$, *where H is a halfplane containing x and minimizing* $\mathrm{Vol}(K \cap H)$
- $w(x)$ ... *width of $C(x)$ in direction orthogonal to the bounding hyperplane of $C(x)$*
- $K_0 = \{x \in K : w(x) \le w_d\}$

Theorem 6. *Fix $D > 1$. For every $K \in \mathcal{K}_D^d$ with* $\mathrm{Vol}K \to \infty$*:*

$$\int_{K_0 \cap K(u \ge 1)} \frac{dx}{u(x)} \ll M(K) \ll \int_K \frac{dx}{1 + u(x)}.$$