Ellis-Filmus-Friedgut: Triangle-intersecting families (presented by Honza Hladky)

Motivation from Extremal Combinatorics: *On an n-element set X, find a big family of sets $\mathcal{F} \subset 2^X$ such that for any $A, B \in \mathcal{F}$: $|A \cap B| \geq 17$.* The first construction is taking $\mathcal{F}$ to be a family of all sets containing fixed 17 elemements of X. This $\mathcal{F}$ has size 2^{n-17}. The Erdős-Ko-Rado asserts that this is optimal.

$\mathcal{G}_n$... all graphs on the vertex set $\{1, \dots, n\}$. We identify graphs with their edges set (in particular, $|G|$ is the number of edges), and further view them as elemements of the group $\mathbb{Z}_2^{\binom{n}{2}}$ (the group operation is xor-ing edges).

•a family $\mathcal{F} \subset \mathcal{G}_n$ is *triangle-intersecting* if $G_1 \cap G_2$ contains a triangle for each $G_1, G_2 \in \mathcal{F}$

•a family $\mathcal{F} \subset \mathcal{G}_n$ is *odd-cycle-intersecting* if $G_1 \cap G_2$ contains an odd cycle for each $G_1, G_2 \in \mathcal{F}$

•a family $\mathcal{F} \subset \mathcal{G}_n$ is *triangle-agreeing* if $\overline{G_1 \triangle G_2}$ contains a triangle for each $G_1, G_2 \in \mathcal{F}$

•a family $\mathcal{F} \subset \mathcal{G}_n$ is *odd-cycle-agreeing* if $\overline{G_1 \triangle G_2}$ contains an odd cycle for each $G_1, G_2 \in \mathcal{F}$

Question (Simonovits-Sós): Construct a big triangle-intersecting family $\mathcal{F} \subset \mathcal{G}_n$.
First construction: Take $\mathcal{F}_0$ to be all graphs containing the triangle 123; $|\mathcal{F}_0| = 2^{\binom{n}{2}-3}$.
Main theorem: $\mathcal{F}_0$ above is optimal.
We shall actually prove that $\mathcal{F}_0$ is optimal for the problem of finding a big odd-cycle agreeing family.

To see that *the agreement property* is not an actual strengthening to *the intersection property* we prove:
Lemma (Chung-Frankl-Graham-Shearer): If $\mathcal{F} \subset \mathcal{G}_n$ is an H-agreeing family then there exists a family $\mathcal{F}'$ of the same size which is H-intersecting.
Proof: Compression.

Proof of the Main Theorem

We identify sets $\mathcal{H} \subset \mathcal{G}_n$ with their indicator functions $\mathcal{H} : \mathcal{G}_n \to \{0, 1\}$.

Observe that $\mathcal{F} \subset \mathcal{G}_n$ is an odd-cycle agreeing family of graphs iff for each $G \in \mathcal{F}$ and B bipartite:

$$G \triangle \overline{B} \notin \mathcal{F} \ . \quad (1)$$

Therefore, $\mathcal{F} \subset \mathcal{G}_n$ is odd-cycle agreeing iff it is an independent set in the Cayley graph on the group $\mathbb{Z}_2^{\binom{n}{2}}$, generated by the set $\{\overline{B} : B \text{ bipartite}\}$. We therefore want to get the bound $\leq 2^{\binom{n}{2}-3}$ on the independence number of this graph.

Fourier analysis on $\mathbb{Z}_2^{\binom{n}{2}}$

Standard basis of the dual group: χ_S, $\chi_S(T) := (-1)^{|S\cap T|}$.

We shall be working a lot with functions $f : \binom{n}{2} \to \mathbb{R}$ such that their eigenfunctions are the standard basis. We then write $\Lambda = (\lambda_G)_{G\in\mathcal{G}_n}$ for the spectrum of such functions. Further, Λ_{min} are the G's with λ_G minimal.

Continuing the Proof

An operator $A : \mathbb{R}\mathcal{G}_n \to \mathbb{R}\mathcal{G}_n$ is an *OCC (=odd-cycle Cayley) operator* if (1) its eigenfunctions are the standard basis, (2) for each odd cycle agreeing family $\mathcal{F}_n$ we have $\mathcal{F}(G) = 1 \Rightarrow A\mathcal{F}(G) = 0$.

Operators A_B and $A_{\mathcal{B}}$: For a bipartite graph $B \in \mathcal{G}_n$ we define $A_B f(G) := f(G\triangle\overline{B})$, and for a distribution $\mathcal{B}$ on bipartite graphs we define $A_{\mathcal{B}} f(G) := \mathbb{E}_{B\sim\mathcal{B}}[f(G\triangle\overline{B})]$.

Lemma 1: $A_{\mathcal{B}}$ is an OCC operation with spectrum $\lambda_R = (-1)^{|R|}\mathbb{E}[\chi_B(R)]$.

Theorem 2 (weighted version of the Hoffman bound): If Λ is an OCC spectrum with $\lambda_0 = 1$ and $\lambda_{min} \in (-1, 0)$ then each odd-cycle agreeing family $\mathcal{F}$ satisfies $\mu(\mathcal{F}) \leq \nu := -\frac{\lambda_{min}}{1-\lambda_{min}}$.

Lemma 3: For each $B \in \mathcal{G}_n$ bipartite, let f_B be an arbitrary function with domain being the subgraphs of B. Let $\mathcal{B}$ be an arbitrary distribution on bipartite graphs. Then the following spectrum is gives an OCC operator:

$$\lambda_G := (-1)^{|G|}\mathbb{E}[f_B(B \cap G)] \; .$$

Define $q_i(G) := \mathbb{P}[|G \cap B| = i]$, where B is a random uniform complete bipartite graph (i.e., partition randomly $\{1, \ldots, n\}$ into two classes, and look at the crossing edges).

Corollary 4 (of Lemma 3): The spectrum $\left(\lambda_G = (-1)^{|G|}q_i(G)\right)_G$ is a spectrum of an OCC operator.

Important Lemma: The spectrum

$$\lambda_G = (-1)^{|G|}\left(q_0(G) - \frac{5}{7}q_1(G) - \frac{1}{7}q_2(G) + \frac{3}{28}q_3(G)\right)$$

is a spectrum of an OCC operator (this is easy) and has the following eigenavalues:

- $\lambda_0 = 1$,
- $\lambda_{min} = -\frac{1}{7}$

Observe that the Main Theorem follows from Theorem 2 and the Important Lemma.

Proof of the Important Lemma

probability generating function $Q_G(X) := \sum_{k \geq 0} q_k X^k$.
graph H.
Lemma A: Let G be an k-vertex graph. Then

1. $q_0(G) = 2^{\#components-k}$,
2. $q_1(G) = \#bridges \times q_0(G)$,
3. if G contains a vertex of odd degree, then: $q_k \leq 1/2$ for each $k \geq 0$,
4. for any odd k, $q_k(G) \leq 1/2$,
5. $q_2(G) \leq 3/4$.

Lemma B: We have $q_0 = 1$, $q_0(K_2) = 1/2$ and $q_0(H) \leq 1/4$ for other graphs.
Lemma C: If $m = 0$ and $|G|$ is odd then $q_0(G) \leq 1/16$, or G is a triangle, or a K_4^-.