# Perfect matchings in $O(n \log n)$ time for regular bipartite graphs

Ashis Goel, Michael Kapralov and Sanjeev Khanna

April 29, 2010

Presented by Bernard Lidický

- graphs are actually multigraphs in this talk

**Theorem 1.** *There exists an $O(n \log n)$ expected time algorithm for finding a perfect matching in a $d$-regular bipartite graph $G = (P, Q, E)$ given in adjacency array representation. ($P, Q$ is the bipartition)*

**Notation:**

$n$ - size of $P$ and as well $Q$
$k$ - number of unmatched vertices in $P$ (and in $Q$)
deg(.) - vector with out degrees
$Z_j$ - expected number of visits of vertex $j$ in a random walk (from $s$ to $t$)
$b_j = 2(1 + \frac{n}{n-j})$ - limit of steps for $j$th augmentation
$X_j$ - number of steps in $j$th augmentation (random variable)
$Y_j$ - "upper bound" on $X_j$ (random variable)
$Y = \sum_{j=0}^{n-1} Y_j$ - "upper bound" on number of steps of perfect matching algorithm
$\mu_j = \frac{b_j}{\ln 2} = E[Y_j]$

**Definition 2.** Let $M$ be a partial matching in $G = (P, Q, E)$. The *matching graph* $H$ is obtained from $G$ by orienting edges from $P$ to $Q$, contracting edges in $M$, adding two new vertices $s$ and $t$ and adding $d$ directed edges from $s$ to every unmatched vertex of $P$ and from every unmatched vertex of $Q$ $d$ directed edges to $t$.

**Lemma 3.** *Expected number of steps before a random walk on $H$ started at $s$ ends at $t$ is at most $1 + n/k$.*

Proof by counting expected number of visits at every vertex during a random walk. Uses transition matrix of the random walk.

**Algorithm TRUNC_RW($b$):**

- "construct" $H$
- start random walk in $s$
- walk until $t$ reached or $b$ steps used
- return $s - t$ path or fail

**Algorithm PERFECT_MATCHING:**

- $j = 0$, $M = \emptyset$
- repeatedly run TRUNC_RW($b_j$) until is succeeds (obtain augmenting path $p$)
- use $p$ on $M$ to get larger matching
- $j := j + 1$
- goto second bullet

**Proof of Theorem 1**

- switch from computing with horrible $X_j$ to nice $Y_j$
- do some estimating
- show that $P[Y \geq cn \log n] \leq n^{-c'}$

**Theorem 4.** *Every deterministic algorithm for finding a perfect matching in d-regular bipartite graph has time complexity* $\Omega(dn)$.

By a game between an algorithm and proof, where algorithm asks for new neighbours of vertex and proof reply with a vertex of his choice. Proof loose one it reveals edges containing a perfect matching.

Exists a graph on $8d + 2$ vertices where game takes $d^2$ steps.