ON THE POWER OF LINEAR DEPENDENCIES

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STEINITZ LEMMA problem: we have d-dimensional Euclidean space $\mathbb{R}^d$, together with norm $||.||$ and unit ball B. Let $V \subset B$ is finite subset of vectors in the unit ball B such that $\sum_{v \in V} v = 0$. We want to find ordering $v_1, v_2, \ldots, v_n$ of vectors from V in such way, that size of all partial sums $\sum_{i=1}^{k} v_i$ (for $k \in [n]$) are bounded by a number that only depends on B.

Theorem 1 *(Steinitz lemma) Given a finite set $V \subset B$ with $\sum_{v \in V} v = 0$, where B is the unit ball of a norm in $\mathbb{R}^d$, there is an ordering $v_1, v_2, \ldots, v_n$ of the elements of V such that for all $k \in [n]$*

$$\sum_{i=1}^{k} v_i \in dB$$

Theorem 2 *Given a finite set $V \subset B_2$ with $\sum_{v \in V} v = 0$, there is an ordering $v_1, v_2, \ldots, v_n$ of the elements of V such that for all $k \in [n]$*

$$\sum_{i=1}^{k} v_i \in \sqrt{\frac{4^d - 1}{3}} B$$

Lets define the smallest constant the Steinitz lemma hold by $S(B)$ (so $S(B) \leq d$ for all B in $\mathbb{R}^d$). For non-symetric norms is $S(B) \leq d$ the best possible. But there are other results for some norms: $S(B_1) \geq \frac{d}{2}$, $S(B_2^2) = \frac{\sqrt{5}}{2}$, $S(B_2^d) \geq \frac{\sqrt{d+3}}{2}$,.. It is conjectured that $S(B_2^d) = O(\sqrt{d})$ and $S(B_\infty^d) = O(\sqrt{d})$, but even much weaker $S(B_2^d) = o(d)$ and $S(B_\infty^d) = o(d)$ estimates seems to be out of reach.

SIGNED SUM problem: $V \subset B$ is a finite set and we want signs $\varepsilon(v) = 1$ or -1 for all $v \in V$, such that $\sum_{v \in V} \varepsilon(v) v$ is not too large.

Theorem 3 *Let $V \subset B_2$ be finite set of vectors, then there are signs $\varepsilon(v)$ such that*

$$\sum_{v \in V} \varepsilon(v) v \in \sqrt{d} B_2$$

The example when V consist of d pairwise orthogonal unit vectors shows that the above estimate is best possible.

SIGNING VECTOR SEQUENCES problem: Let U be sequence (finite or infinite of vectors $u_1, u_2, \dots$ from the symmetric unit ball $B \subset \mathbb{R}^d$. We wish to finds signs $\varepsilon_i \in \{+1, -1\}$ such that all partial sums $\sum_{i=1}^n \varepsilon_i u_i$ are bounded by a constant depending only on B.

Theorem 4 *Under the above conditions there are signs ε_i such that for all $n \in \mathbb{N}$*

$$\sum_{i=1}^{n} \varepsilon_i u_i \in 2dB$$

Define the sign sequence constant, $E(B)$, of unit ball B is the smallest blow up factor for which previous theorem hold. The methods for the Steinitz constant $S(B)$ and the sign sequence constant $E(B)$ are similar, and so are bounds.

Theorem 5 *Assume B is the unit ball of symmetric norm in $\mathbb{R}^d$. Then*

$$S(B) \leq E(B)$$