Intersection Patterns of Curves

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Definitions

- A family $\mathcal{F}$ of finite graphs has the *Erdős-Hajnal property* if there is a constant $c(\mathcal{F}) > 0$ such that every graph in $\mathcal{F}$ on n vertices contains a clique or an independent set of size at least $n^{c(\mathcal{F})}$.
- *bi-clique*: a complete bipartite graph whose vertex classes differ in size by at most one
- A family $\mathcal{F}$ of finite graphs has the *strong Erdős-Hajnal property* if there is a constant $b(\mathcal{F}) > 0$ such that for every graph $G \in \mathcal{F}$ on $n > 1$ vertices, G or its complement $\overline{G}$ contains a bi-clique of size at least $b(\mathcal{F})n$.

Observation [Alon, Pach, Pinchasi, Radoičić, Sharir, 2005]. For a hereditary family $\mathcal{F}$, the Strong Erdős-Hajnal property implies the Erdős-Hajnal property.

Theorem [Alon et al., 2005]. Intersection graphs of semialgebraic sets of constant description complexity in $\mathbb{R}^d$ have the Strong Erdős-Hajnal property.

Main Theorem. Let C be a k-intersecting collection of $n > 1$ curves in the plane such that the number of intersecting pairs of curves is at least εn^2, for some $\varepsilon > 0$. Then the intersection graph of C contains a bi-clique of size at least $c_k \varepsilon^{64} n$, where $c_k > 0$ is a constant that depends only on k.

Corollary. For every $k \in \mathbb{N}$, the family of intersection graphs of k-intersecting collections of curves in the plane has the strong Erdős-Hajnal property.

Proof of the Main Theorem

- *comparability graph* of a poset $(P, <)$ is a graph $(P, < \cup >)$
- *incomparability graph* of a poset $(P, <)$ is a complement of the comparability graph of $(P, <)$
- *permutation graph* is a comparability graph of a two-dimensional poset (intersection of two linear orders)
- *interval graph* is an intersection graph of subintervals of $\mathbb{R}$
- $[0, 1]$-*curve* is a curve with endpoints on the lines $L_1 : x = 0$ and $L_2 : x = 1$ contained in the strip between L_1 and L_2

Observation. The following three statements are equivalent:
(i) G is an incomparability graph.
(ii) G is the intersection graph of a collection of $[0, 1]$-curves.
(iii) G is the intersection graph of a collection of x-monotone $[0, 1]$-curves.

Lemma PERM. If G is a $K_{t,t}$-free permutation graph with n vertices, then G is $(2t-2)$-degenerate. In particular, G has at most $(2t-2)n - \binom{2t-1}{2}$ edges.

Lemma coINT. If G is a $K_{t,t}$-free complement of an interval graph with n elements, then G has at most $2(t-1)n$ edges.

Lemma INC. If G is a $K_{t,t}$-free incomparability graph with n vertices and $m \geq \sqrt{5tn^3}$ edges, then G contains an induced subgraph with at least $m^2/5n^3$ vertices and edge density at least $1 - 10tn^3/m^2$.

Lemma 1. Let k be a positive integer, let A a be collection of *double-grounded curves* with grounds γ_1 and γ_2, and let B be a collection of curves such that each curve in B intersects γ_1 and γ_2 in at most k points. If, for some $d \geq 6$,
(i) each curve in A intersects at most d other curves in A, and
(ii) each curve of B intersects at least $15kd$ curves in A,
then there are subsets $A' \subset A$ and $B' \subset B$ of size $|A'| \geq d/3$ and $|B'| \geq \frac{d|B|}{3|A|}$ such that every curve in A' intersects every curve in B'.

Lemma 2. Let $k \geq 1$ and $m \geq 144$ be integers, let A be a collection of double-grounded curves with grounds γ_1 and γ_2, and let B be a collection of curves such that every curve in B intersects γ_1 and γ_2 in at most k points. If
(i) there are at most m intersecting pairs in A, and
(ii) there are at least $20k\sqrt{m}|B|$ intersecting pairs in $A \times B$, then there are subsets $A' \subset A$ and $B' \subset B$ of size $|A'| \geq \sqrt{m}/7$ and $|B'| \geq km|B|/|A|^2$ such that every curve in A' intersects every curve in B'.

Theorem 3. (Main Theorem for $[0,1]$-curves)
Given a k-intersecting collection C of n $[0,1]$-curves with at least εn^2 intersecting pairs, its intersection graph contains a bi-clique of size at least $c_k\varepsilon^2 n$, where $c_k > 0$ depends only on k.

proof: by induction on k
$f(\varepsilon, \delta, k, n) :=$ the largest integer $t \in \mathbb{N}$ such that for any collection C of n $[0,1]$-curves in the plane in general position with at least εn^2 intersecting pairs, at most δn^2 pairs of which intersect in more than k points, the intersection graph of C contains $K_{t,t}$.

Lemma 3.1. (base case of the induction)

$$f\left(\varepsilon, \frac{\varepsilon}{2}, 1, n\right) \geq \frac{\varepsilon}{4} n.$$

Theorem 3.k. (induction step)
For all $\varepsilon, \delta > 0$ with $4000\delta \leq \varepsilon^4$, $k \geq 2$, $n \geq 2$, and $t \leq \frac{\varepsilon^2 n}{10^7 k}$, we have

$$f(\varepsilon, \delta, k, n) \geq \min(t, f(\varepsilon', \delta', k-1, n')),$$

where $\varepsilon' = \frac{1}{10^6 k^2}$, $\delta' = 10^6(\frac{\delta}{\varepsilon^4} + \frac{t}{\varepsilon^2 n})$, and $n' \geq \frac{\varepsilon^2 n}{250}$.

Theorem 4. (Main Theorem for grounded curves)
Let C be a k-intersecting collection of n curves. Suppose that $C_1 \subseteq C$ is a collection of grounded curves with ground g, $C_2 = C \setminus C_1$ is a collection of curves disjoint from g, and there are at least εn^2 intersecting (nonidentical) pairs in $C_1 \times C$.
(i) If $C = C_1$ (hence $C_2 = \emptyset$), then the intersection graph of C contains a bi-clique of size at least $ck\varepsilon^8 n$, where $c_k > 0$ is a constant depending on k only.
(ii) Otherwise, the intersection graph of C contains a bi-clique of size at least $c'_k\varepsilon^{32} n$, where $c'_k > 0$ is a constant depending on k only.