On the measure of intersecting families, uniqueness and stability

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A family $\mathcal{A} \subseteq [n]$ is *intersecting* if the intersection of every two members of $\mathcal{A}$ is nonempty; it is *t-intersecting* if such an intersection is of size at least t. For fixed k and $A \subset [n]$ the *principal family defined by A* is the family of all k-subsets of $[n]$ containing A.

A family $\mathcal{A} \subseteq [n]$ is identified with its characteristic vector from $\{0,1\}^n$. A function $f : \{0,1\}^n \to \{0,1\}$ is *dictatorship* if there is $i \in [n]$ such that $f(x) = x_i$; it is t-umvirate if there exists a set $A \subseteq [n]$ with $|A| = t$ and $f(x) = \prod_{i \in A} x_i$.

For every $p \in [0,1]$ let $q = 1 - p$, then μ_p is the product measure on $\{0,1\}^n$ given by $\mu_p(x) = p^{\alpha}(x)q^{\beta}(x)$ where α is the number of 1s of x and β is the number of 0s of x.

Theorem (Erdős-Ko-Rado). *Let $k \leq n/2$ and let $\mathcal{A} \subseteq \binom{[n]}{k}$ be an intersecting family. Then $|\mathcal{A}| \leq \binom{n-1}{k-1}$. Furthermore, if $k < n/2$ then equality is attained if and only if $\mathcal{A}$ is a principal family defined by some $\{i\}$.*

Theorem (Fishburn & al). *Let $0 \leq p \leq 1/2$ and let $\mathcal{A} \subset \{0,1\}^n$ be an intersecting family. Then $\mu_p \leq p$. Furthermore, if $0 < p < 1/2$ then equality is attained if and only if $\mathcal{A}$ is dictatorship.*

Theorem 1. *Let $0 < p < 1/2$ and let $\mathcal{A} \subset \{0,1\}^n$ be an intersecting family. Then*

1. $\mu_p(\mathcal{A}) \leq p$.
2. **Uniqueness:** *If $\mu_p(\mathcal{A}) = p$ then $\mathcal{A}$ is dictatorship.*
3. **Stability:** *If $\mu_p(\mathcal{A}) = p - \varepsilon$ then there exists dictatorship $\mathcal{B}$ such that $\mu_p(\mathcal{A} \triangle \mathcal{B}) = c\varepsilon$ where $c = c(p)$.*

Corollary 2. *Let $0 < \zeta$, let $\zeta n < k < (\frac{1}{2} - \zeta)n$ and let $\mathcal{A} \subseteq \binom{[n]}{k}$ be an intersecting family. If $|A| \geq (1-\varepsilon)\binom{n-1}{k-1}$ then there exists a principal family $\mathcal{B} \subset \binom{[n]}{k}$ defined by some $\{i\}$ such that $|\mathcal{A} \setminus \mathcal{B}| < c\varepsilon\binom{n}{k}$ where $c = c(\zeta)$.*

Few more definitions:

$$M(n,k,t) = \max\left\{|\mathcal{A}| : \mathcal{A} \subset \binom{[n]}{k}, \mathcal{A} \text{ is } t\text{-intersecting}\right\}.$$

$$I(n,k,t,r) = \left\{A \in \binom{[n]}{k}, |A \cap [t+2r]| \geq t + r\right\}.$$

$I(n,k,t,r)$ is a t-intersecting family.

Theorem (Alshwede-Khachatrian). $M(n,k,l) = \max_r\{|I(n,k,t,r)|\}$.

Theorem (Dinur-Safra)**.** *Let* $0 < p < \frac{1}{2}$*, let* $1 \leq t$ *and let* $\mathcal{A} \subseteq \{0,1\}^n$ *be a* t*-intersecting family. Then* $\mu_p(\mathcal{A}) \leq \max_r \mu_p(\{x : \sum_{i=1}^{k+2r} x_i \geq t + r\})$*.*

Theorem 3. *Let* $t \geq 1$ *be an integer, let* $0 < p < \frac{1}{t+1}$ *and let* $\mathcal{A} \subset \{0,1\}^n$ *be a* t*-intersecting family. Then*

1. $\mu_p(\mathcal{A}) \leq p^t$.
2. **Uniqueness:** *If* $\mu_p(\mathcal{A}) = p^t$ *then* $\mathcal{A}$ *is* t*-umvirate.*
3. **Stability:** *If* $\mu_p(\mathcal{A}) = p^t - \varepsilon$ *then there exists* t*-umvirate* $\mathcal{B}$ *such that* $\mu_p(\mathcal{A} \triangle \mathcal{B}) = c\varepsilon$ *where* $c = c(p)$*.*

Corollary 4. *Let* $t \geq 1$ *be an integer, let* $0 < \zeta, \zeta n < k < (\frac{1}{t+1} - \zeta)n$ *and let* $\mathcal{A} \subset \binom{[n]}{k}$ *be a* t*-intersecting family. If* $|A| \geq (1-\varepsilon)\binom{n-t}{k-t}$ *then there exists a principal family* $\mathcal{B} \subset \binom{[n]}{k}$ *defined by some* B *with* $|B| = t$ *such that* $|\mathcal{A} \setminus \mathcal{B}| < c\varepsilon\binom{n}{k}$ *where* $c = c(\zeta)$*.*

Theorem (Hoffman)**.** *Let* G *be a regular graph with eigenvalues* $\lambda_1 \geq \cdots \geq \lambda_m$*. Then*

$$\bar{\alpha}(G) \leq \frac{-\lambda_m}{\lambda_1 - \lambda_m}$$

where $\bar{\alpha}(G)$ *is the weighted size of the largest independent set, i.e.,* $\bar{\alpha}(G) = \frac{\alpha(G)}{m}$*.*

A few important matrices

$$A^{(1)} = \begin{pmatrix} \frac{q-p}{q} & \frac{p}{q} \\ 1 & 0 \end{pmatrix} \qquad \text{eigenvectors: } \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \sqrt{p/q} \\ -\sqrt{q/p} \end{pmatrix} \qquad \text{eigenvalues: } 1, -p/q;$$

2-disjointness matrices; the rows and twe columns of $B^{(3)}$ and $M^{(3)}$ are indexed with $\emptyset, \{1\}, \{2\}, \{1,2\}, \{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}$:

$$B^{(3)} = \begin{pmatrix} 1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1 \\ 1&1&1&0&1&1&1&0 \\ 1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&0&1&0 \\ 1&1&1&1&1&1&0&0 \\ 1&1&1&0&1&0&0&0 \end{pmatrix}; M^{(1)} = \begin{pmatrix} 1 & 1 \\ 1 & X \end{pmatrix}; M^{(3)} = \begin{pmatrix} 1&1&1&1&1&1&1&1 \\ 1&X&1&X&1&X&1&X \\ 1&1&X&X&1&1&X&X \\ 1&X&X&X^2&1&X&X&X^2 \\ 1&1&1&1&X&X&X&X \\ 1&X&1&X&X&X^2&X&X^2 \\ 1&1&X&X&X&X&X^2&X^2 \\ 1&X&X&X^2&X&X^2&X^2&X^3 \end{pmatrix};$$

$$D^{(1)} = \begin{pmatrix} \frac{q-p}{q} + \frac{p}{q}X & \frac{p}{q} - \frac{p}{q}X \\ 1 - X & X \end{pmatrix} \quad \text{eigenvectors: } \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \sqrt{p/q} \\ -\sqrt{q/p} \end{pmatrix} \quad \text{eigenvalues: } 1, -p/q(1-X/p);$$