Large almost monochromatic subsets in hypergraphs

David Conlon, Jacob Fox and Benny Sudakov

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Presented by Bernard Lidický

Notation:

$r_k(n,l)$: minimum size of a set N such that for every coloring of k-tuples from N by l colors exists $M \subset N$ of size n such that all k-tuples on M are colored the same

$r(H,l)$: minimum size of a set N such that for every coloring of k-tuples from N by l colors exists a monochromatic copy of H, where H is a k-uniform hypergraph

$K_d^k(n)$: k-uniform hypergraph whose vertex set consists of d parts of size n and whose edges are all k-tuples with vertices in k different parts

Message:

The gap between sizes of monochromatic and almost monochromatic subsets can be large for triples. It is known that for ≥ 4 colors, the largest monochromatic subset has cardinality $\Theta(\log\log N)$, while the almost monochromatic is $\Theta(\sqrt{\log N})$ due to Theorem 1.

Erdős doubts his conjecture that $r_3(n,2)$ is close to $2^{2^{cn}}$ (maybe closer to 2^{cn^2}).

Theorem 1. *For each $\epsilon > 0$ and l, there is $c = c(l,\epsilon) > 0$ such that every l-coloring of the triples of an N-element set contains a subset S of size $s = c\sqrt{\log N}$ such that at least $(1-\epsilon)\binom{s}{3}$ 3 triples of S have the same color.*

- corollary of the following theorem

Theorem 2. *The l-color Ramsey number of the complete d-partite hypergraph $K_d^3(n)$ satisfies*

$$r(K_d^3(n);l) \leq 2^{l^{2r}n^2},$$

where $r = r_2(d-1;l)$.

Lemma 3. *Let G be a bipartite graph with parts A and B and with at least $|A||B|/l$ edges. Then G contains a complete bipartite subgraph with one part having $a = |A|/l$ vertices from A and the other part having $b = 2^{-|A|}|B|$ vertices from B.*

- counting couples and pigeonhole

Lemma 4. *If a graph G of order n has ϵn^2 edges and $t < \epsilon n$, then it contains $K_{s,t}$ with $s = \epsilon^t n$.*

- counting couples

Proof of Theorem 2

- set $r = r_2(d-1,l)$, $N = 2^{l^{2r}n^2}$
- construct $V_1, \ldots, V_{r+1}$, each of size n such that for every $1 \leq i < j \leq r$ all triples in $V_i \times V_j \times V_k$, where $j < k \leq r+1$ have the same color $\chi(i,j)$
- find monochromatic subgraph of $d-1$ sets V_i and add V_{r+1}.