Economical elimination of cycles in the torus

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In this talk, all intervals are intervals of integers. For example, $[-m; m) = \{-m, -m+1, ..., m-1\}$.

Let $G(m,d)$ denote the graph on the vertex set $[-m; m)^d$. Two vertices x and y of $G(m,d)$ are adjacent, if there exists some $i \in [1; d]$ such that $x_j = y_j$ for $j \neq i$ and $x_i \equiv y_i \pm 1 (\bmod 2m)$. I.e. $G(m,d)$ is the graph of the d-dimensional torus.

A cycle in $G(m,d)$ is called *nontrivial* if it wraps arround the torus. In particular, every projection of the cycle to any coordinate contains all $2m$ possible vertices.

A set of vertices of $G(m,d)$ is called a *spine* if every nontrivial cycle has an non-empty intersection with this set.

Theorem 1. *There exists an (absolute) constant c such that for every m and every $d \geq 2$ the graph $G(m,d)$ contains a spine of at most $c\frac{\log d}{m}n$ vertices, where $n = (2m)^d$ is the number of vertices of $G(m,d)$.*

For any set X of vertices let $N(X)$ be the set of all neighbours of X not contained in X. Moreover, let

$$N^+(X) = N(X) \cup X.$$

Lemma 2 (main)**.** *There exists a set B of vertices of $G(m,d)$ which does not contain a nontrivial cycle satisfying*

$$\frac{|N(B)|}{|B| + |N(B)|} \leq c\frac{\log d}{m}$$

Sketch of the proof of lemma 2. Take B as the ℓ_1-ball of radius $x = \frac{md}{5\log d}$ intersected with the torus and then circuitously calculate that B has the expected property. □

Sketch of the proof of theorem 1.

- Take such B as in Lemma 2 and take random shifts of B around the torus. These random shifts let be denoted by $B_1, B_2, ...$
- With probability 1 there exists some integer s such that the random shifts $B_1, ..., B_s$ covers the torus.
- Take sets $S_i = N^+(B_i) \setminus \bigcup_{j<i} N^+(B_j)$ and let $S = \bigcup_{i=1}^{s} S_i$. Then S is always an spine.
- By a simple averaging argument show, that the expected size of S is small.

□