Anti-Ramsey Property of Graphs

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With two basic notion

1. *b-bounded coloring* as coloring of $E(G)$ s.t. no color is useded more than b times and
2. *rainbow* as subset of $E(G)$ in which each edge has different color,

the main graph property under consideration is defined:

$$G \in \mathcal{A}(b, H) \quad \text{iff} \quad \text{every } b\text{-bounded coloring of } E(G) \text{ has a rainbow copy of H}$$

The article is searching for threshold for random graph $G_{n,p}$ to have property $\mathcal{A}(b, H)$.

Other notation

$$m_H = \frac{e_H - 1}{v_h - 2} \qquad m^*_H = \max\{m_{H'} : H' \subseteq H, v_H \geq 3\}$$
$$p^* = \frac{1}{n^{1/m^*_H}}$$

Theorem 1. *For all graphs H containing at least one cycle there exists a constant $b_0 = b_0(H)$ such that if $b \geq b_0$ then there exist $c_1 = c_1(b, H)$ and $c_2 = c_2(b, H)$ such that if $p = cn^{-1/m^*_H}$ then*

$$\lim_{n\to\infty} \mathbf{Pr}(G_{n,p} \in \mathcal{A}(b, H)) = \begin{cases} 0 & \text{if } c \leq c_1 \\ 1 & \text{if } c \geq c_2 \end{cases}$$

Theorem 2. *Let $p = \frac{c_n}{n^{2/3}}$. Then*

$$\lim_{n\to\infty} \mathbf{Pr}(G_{n,p} \in \mathcal{A}(2, K_3)) = \begin{cases} 0 & \text{if } c_n \to 0 \\ 1 - \exp^{-c^6/24} & \text{if } c \to c \\ 1 & \text{if } c \to \infty \end{cases}$$

Theorem 3. *Let $p = \frac{c}{n^{1/2}}$. Then*

$$\lim_{n\to\infty} \mathbf{Pr}(G_{n,p} \in \mathcal{A}(3, K_3)) = \begin{cases} 1 - \exp^{-c^{10}/120} & \text{if } c < 1/\sqrt{2} \\ 1 & \text{if } c > /\sqrt{2} \end{cases}$$

Lemma 6. $K_{r+2} \in \mathcal{A}(r, K_3)$ *for* $r \geq 1$

$$\lim_{n\to\infty} \mathbf{Pr}(G_{n,p} \in \mathcal{A}(2, K_3) \mid G_{n,p} \text{ is } K_4 - \text{free}) = 0 \tag{2}$$

Lemma 7. *Let* Γ *be the triangle graph of* $G_{n,p}$ *with* $p = c/n^{2/3}$ *where* c *is constant.* ***Whp*** *every component* C *of* Γ *satisfy one of the following two conditions*

1. G_C *is isomorphic to* K_4*, or*
2. G_C *is* 2*-degenerate.*

$$\lim_{n\to\infty} \mathbf{Pr}(G_{n,p} \in \mathcal{A}(3, K_3) \mid G_{n,p} \text{ is } K_5 - \text{free}) = 0 \tag{3}$$

Lemma 8. ***Whp*** *every connected component* C *of* Γ *is safe.*

Lemma 9. ***Whp*** *every connected component* C *of* Γ_H *is safe.*