Polynomial, rational, exponential and logarithmic functions

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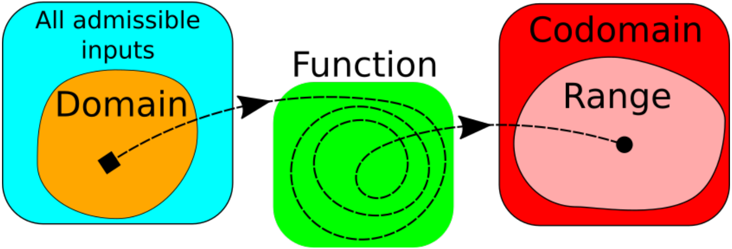
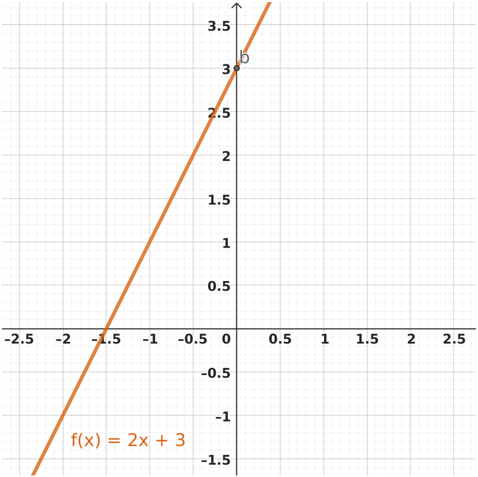
Functions



- Denote the function by the letter f.
- Domain of f = a set of allowed inputs for f.
- Range of f = the set of outputs when evaluated on the domain.
- Co-domain of f = a set containing the range, possibly larger.
- We write a function as $f : X \to Y$.
  X is the domain.
  Y is the co-domain of f.

Linear function

- General form: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = mx + b$.
- Example: $f(x) = 2x + 3$.



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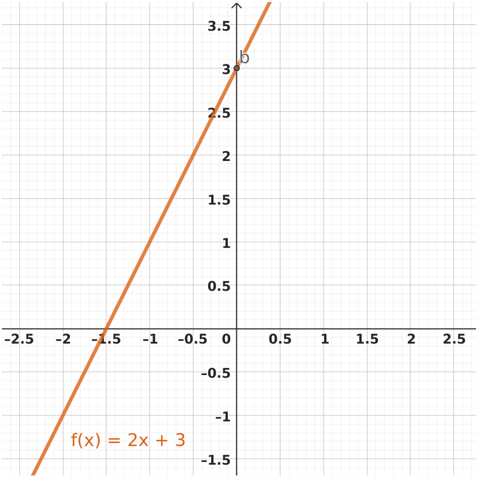

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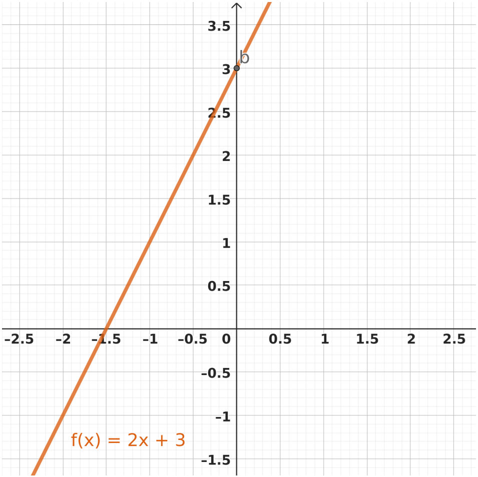

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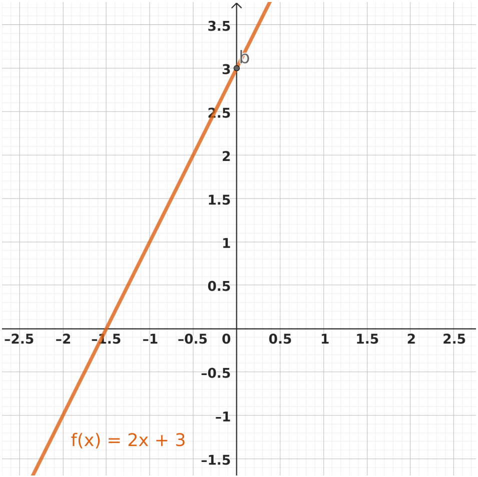
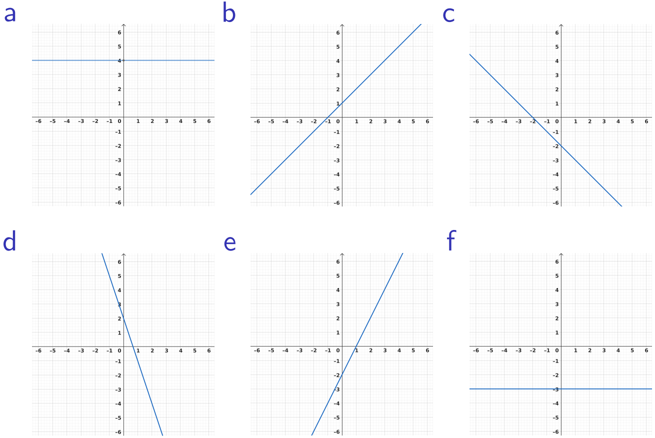


- Domain of $f = \mathbb{R}$.
- Range of $f = \mathbb{R}$.
- Co-domain of $f = \mathbb{R}$.

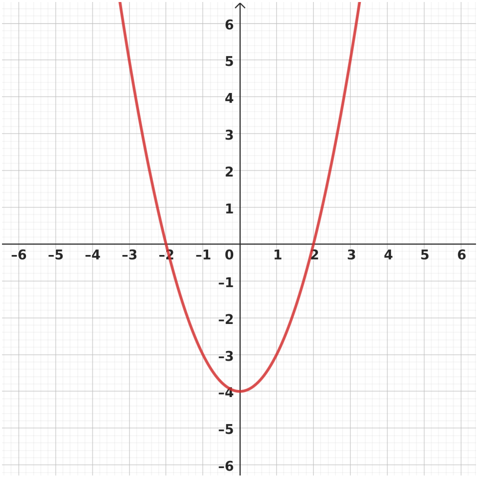
Match linear functions

1. $y = -5.$
2. $y = 2x - 2.$
3. $y = -x - 2.$
4. $y = 2 - 3x.$
5. $y = 4.$
6. $y = x + 1$



Quadratic function

- General form: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = ax^2 + bx + c$.
- Example: $f(x) = x^2 - 4$.



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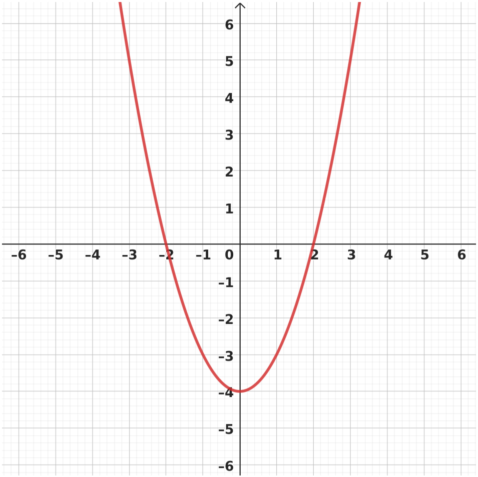

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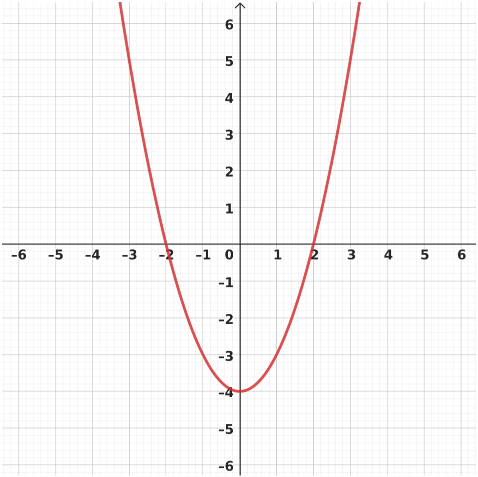

- Domain of $f = \mathbb{R}$.
- Range of $f = [-4, \infty)$.
- Co-domain of $f =$

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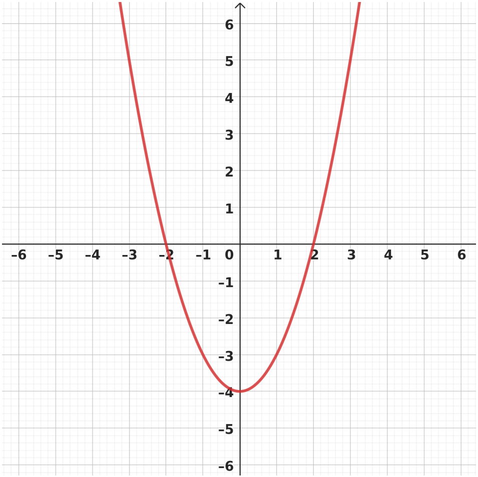
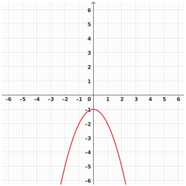


- Domain of $f = \mathbb{R}$.
- Range of $f = [-4, \infty)$.
- Co-domain of $f = \mathbb{R}$.
- We could also write $f: \mathbb{R} \to [-4, \infty)$.

Match quadratic functions

1. $y = x^2$
2. $y = -x^2.$
3. $y = 2x^2 + 3.$
4. $y = 3x^2 - 5.$
5. $y = 2 - 2x^2.$
6. $y = -x^2 - 1.$

a



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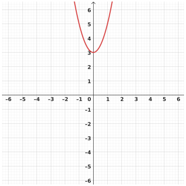
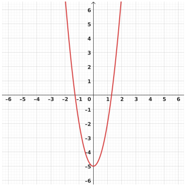
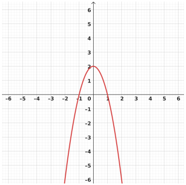


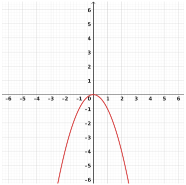
c



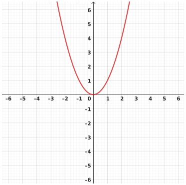
d



e



f



Polynomials

General form of a polynomial of degree d:

$$p(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0 \text{ with } a_d \neq 0.$$

Dictionary:

- x is called variable.
- d is the degree of the polynomial.
- Coefficients: $a_0, \ldots, a_d$.
- Leading coefficient: a_d.
- Monomial: coefficient and variable $a_i x^i$.
- Polynomial: sum of monomials.
- Root: solution to $p(x) = 0$, that is where the graph intersects the horizontal axis.

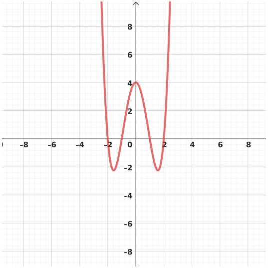
Example

x^2 has one root, $x^2 + 1$ has no real roots, $x^2 - 1$ has two roots

Match polynomials

1. $y = x^3 - x$
2. $y = -x^3 - 5x^2 - 2x + 8.$
3. $y = x^4 - 5x^2 + 4.$
4. $y = -x^4 - 4x^3 + 4x^2 + 16x.$

a



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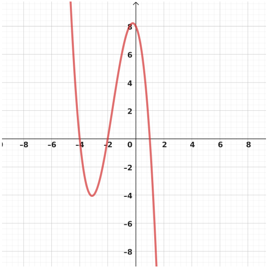

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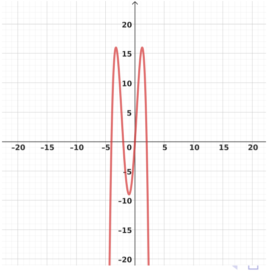

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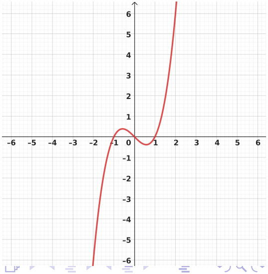
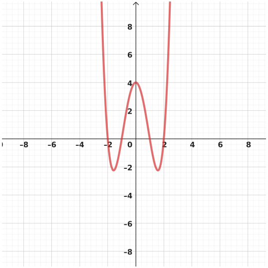


Match polynomials (easier version)

- $y = (x-1)(x+1)(x-2)(x+2)$.
- $y = -x(x-2)(x+2)(x+4)$.
- $y = x(x+1)(x-1)$
- $y = -(x+2)(x+4)(x-1)$.

a



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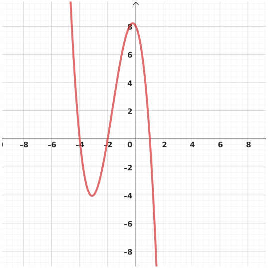

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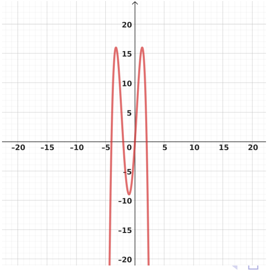

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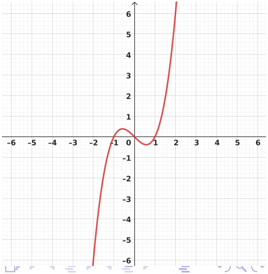
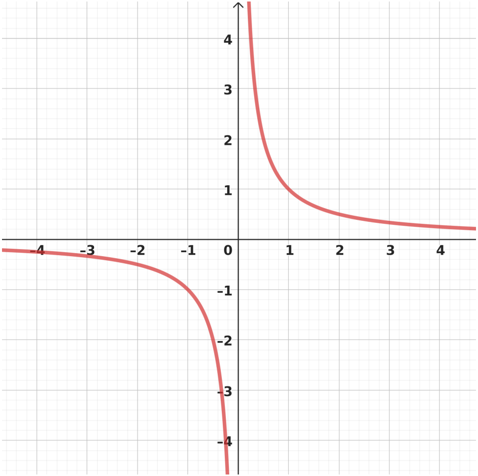


Operations with polynomials

- Addition, subtraction, multiplication.
- Division, not necessary a polynomial.
- Example: divide $x^3 + 3x^2 - 4x - 12$ by $x^2 + x - 6$.
- Let P, Q polynomials of degree n and m, then what is the degree of $P + Q$, $P - Q$, $P * Q$? If P/Q is a polynomial, what is its degree?

Rational functions.

- General form: $P(x)$, $Q(x)$ polynomials then $f : \mathbb{R} \setminus \{x \in \mathbb{R} : Q(x) = 0\} \to \mathbb{R}$, $f(x) = P(x)/Q(x)$.
- Example: $f(x) = \frac{1}{x}$.



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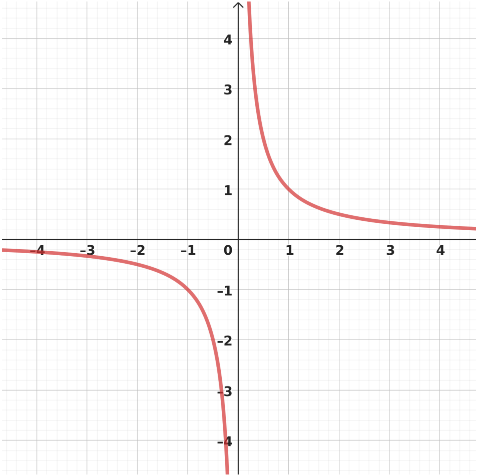

- Domain of $f = \mathbb{R}\setminus\{0\}$
- Range of $f =$

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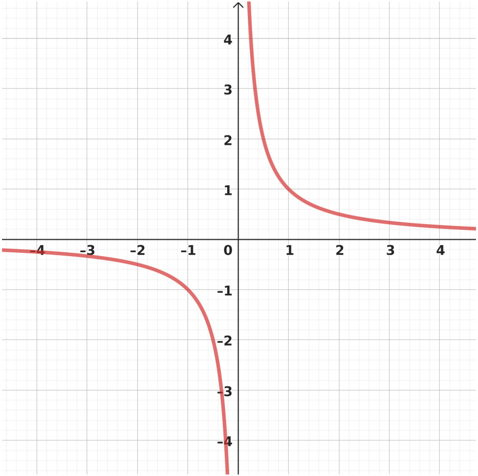

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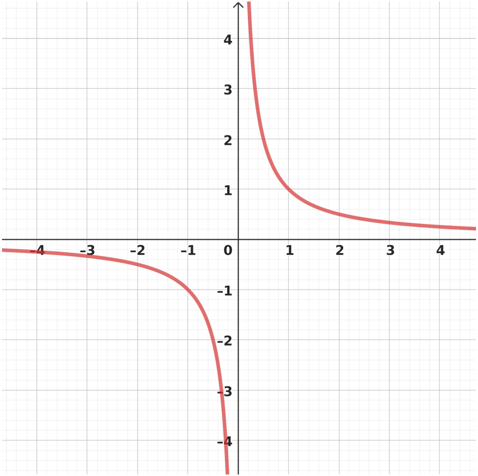
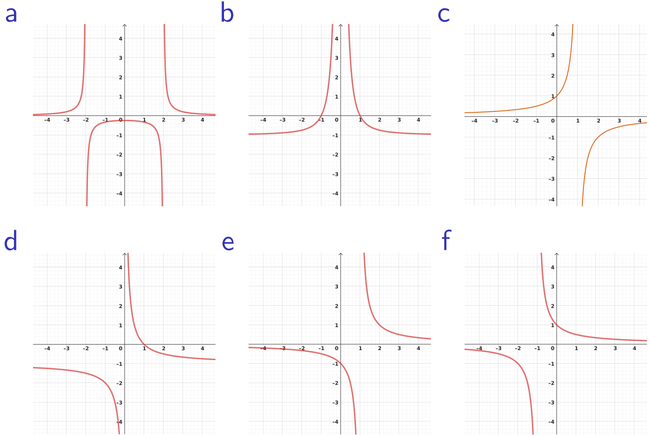


- Domain of $f = \mathbb{R} \setminus \{0\}$
- Range of $f = \mathbb{R} \setminus \{0\}$
- Co-domain of $f = \mathbb{R}$.
- It is not correct to write: $f\colon \mathbb{R} \to \mathbb{R}$.

Match quotient functions

1. $y = \frac{1}{x+1}$.
2. $y = \frac{1-x^2}{x^2}$.
3. $y = \frac{1}{x-1}$.
4. $y = \frac{1}{1-x}$.
5. $y = -\frac{1-x}{x}$.
6. $y = \frac{1}{x^2-2}$



Exponential function.

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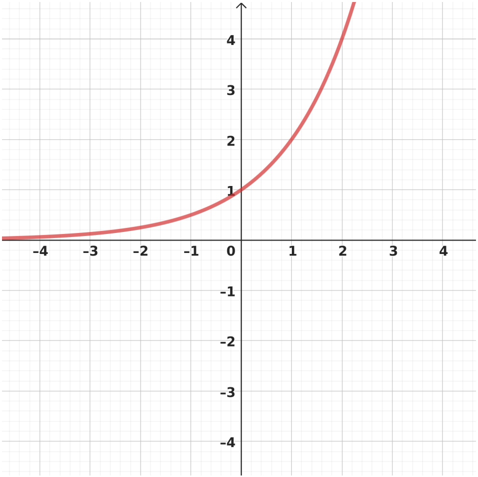

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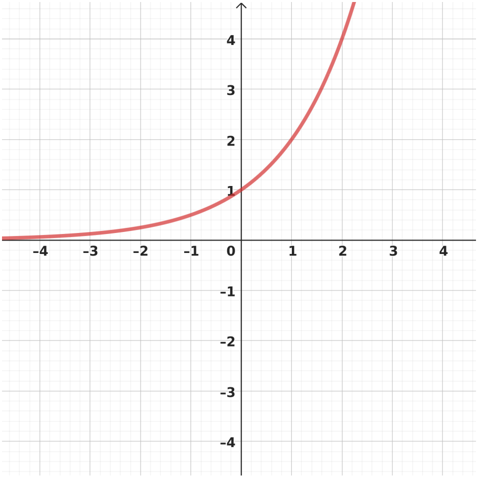

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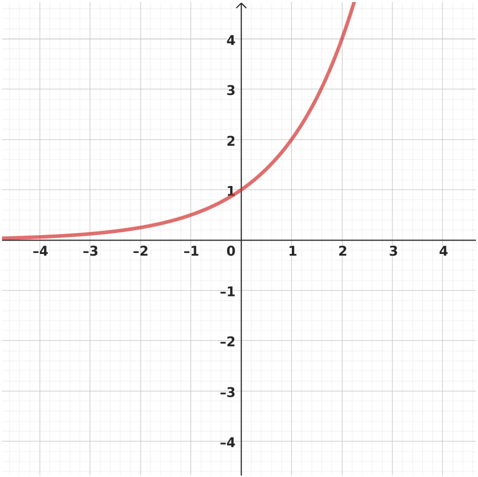

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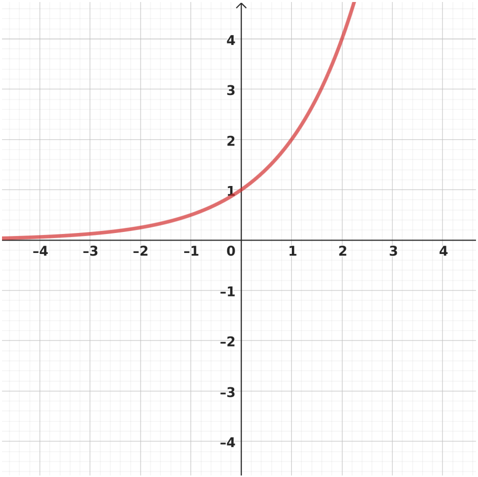

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Properties of functions

- Injective: every value in range is obtained exactly once.
- Surjective: every value of co-domain is obtained.
- Increasing/non-decreasing.
- Decreasing/non-increasing.
- Bounded from above/below.
- Even $f(x) = f(-x)$/ odd $f(-x) = -f(x)$.
- Periodic $f(x) = f(x + p)$ for some period p.

Composition of functions

- $f : X \to Y$ and $g : Y \to Z$, then their composition is $h : X \to Z$ as $h(x) = g(f(x))$.
- We denote it as $h = g \circ f$.
- It is associative.

Example

- $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x + 1$.
- $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$.
- $h(x) = g(f(x)) = (x + 1)^2 = x^2 + 2x + 1$.
- The order is important: $f(g(x)) = x^2 + 1 \neq x^2 + 2x + 1 = g(f(x))$.

To think:

If f and g satisfy one of the properties in the previous slide, is it true that their composition satisfy it as well?

Inverse of a function

- $f: X \to Y$ is invertible if there exists a function $g: Y \to X$ such that $(f \circ g)(y) = y \in Y$ for every y and $(g \circ f)(x) = x$ for every $x \in X$.
- We denote its inverse by $f^{-1}: Y \to X$.
- Domain of f^{-1} = range of f.
- Range of f^{-1} = domain of f.
- Not always possible: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 1$ or $f(x) = x^2$.

Example

- x^2 has inverse on $[0, +\infty)$ that is $\sqrt{(}x)$.
- x^3 has inverse on $\mathbb{R}$ that is $x^{1/3}$.
- $1/x$ is its own inverse.

To think:

The inverse function, if exists, is it unique?

If $f: X \to Y$ and $g: Y \to Z$ are invertible, is $g \circ f$ invertible?

Examples of inverse function

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5. $\ln(x)$ is the inverse function of e^x. So $e^{\ln(x)} = x$ and $\ln(e^x) = x$.

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5. $\ln(x)$ is the inverse function of e^x. So $e^{\ln(x)} = x$ and $\ln(e^x) = x$.
6. In general, $\log_b(y)$ is the inverse function of $f(x) = b^x$.

More examples of exponential and logarithmic functions

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6. Say you invest 1000 USD and you get 5% interest per year. How much money will you have at the end of the n-th year?

Operations with exponential and logarithm

- Exponential: $f(x) = b^x$, b is called base $b > 0$ and $b \neq 1$, x is the exponent.
- Natural base, $e \approx 2.718\ldots$. Irrational, Euler number.
- Arithmetic: $b^x * b^y = b^{x+y}$, $(b^x)^y = b^{xy} \neq b^{x^y}$. $b^0 = 1$ and $\frac{b^x}{b^y} = b^{x-y}$.

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- Exponential: $f(x) = b^x$, b is called base $b > 0$ and $b \neq 1$, x is the exponent.
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- Logarithm: inverse function of exponential , if $y = b^x$, then $log_b(y) = x$.
- Domain is $(0, \infty)$ and range $\mathbb{R}$.
- Natural logarithm: base $b = e$, denote $\ln x = log_e(x)$.
- Fill: $log_b(b)$, $log_b(1)$, $log_b(b^x)$, $b^{log_b(x)}$.

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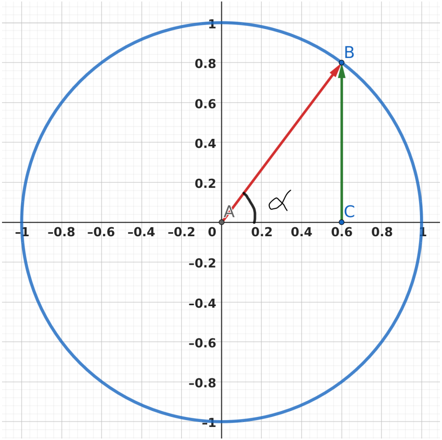
- Arithmetics:
  - $log_b(x * y) = log_b(x) + log_b(y)$.
  - $log_b(x/y) = log_b(x) - log_b(y)$.
  - $log_b(x^n) = nlog_b(x)$.
  - Change of basis: $log_y(x) = \frac{log_b(x)}{log_b(y)}$.

  To do: verify these equalities.

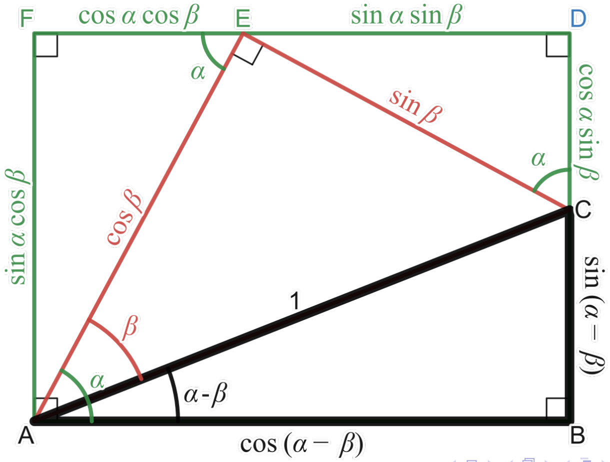
Trigonometric functions



- $\sin(\alpha) = |BC|/|AB|$.
- Domain of $\sin = [0, 2\pi]$.
- Range $\sin = [-1, 1]$.
- $\cos(\alpha) = |AC|/|AB|$.
- Domain $\alpha = [0, 2\pi]$
- Range $\cos = [-1, 1]$.
- $\tan(\alpha) = |BC|/|AC|$.
- Domain $\tan = [0, 2\pi] \setminus \{\pi/2, 3\pi/2\}$.
- Range $\tan = \mathbb{R}$.

Properties of trigonometric functions

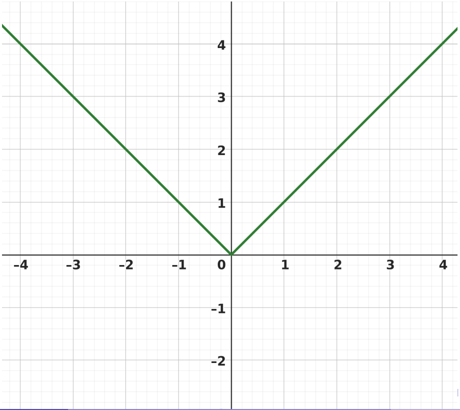
- $\sin(\alpha)^2 + \cos(\alpha)^2 = 1$.
- $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$.
- $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$



Piecewise functions

- $f : X \to Y$ and $X = X_1 \sqcup \cdots \sqcup X_d$, $g_i : X_i \to Y$ and $f(x) = g_i(x)$ if $x \in X_i$.
- Example: absolute value $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$



Recursive functions

- Over the natural number.
- Sketch: $f(n+k) = f(n) + \cdots + f(n+k-1)$.
- Example: $f(0) = 1$ and for $n \geq 1$ set $f(n) = n * f(n-1)$.
- Example: $f(0) = 1$ and for $n \geq 1$ set $f(n) = 3 * f(n-1)$. Closed formula: $f(n) = 3^n$.
- Example: $f(1) = 1$, $f(2) = 1$, and for $n \geq 3$ set $f(n) = f(n-1) + f(n-2)$. Closed formula: $f(n) = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}}$ where $\phi = \frac{1+\sqrt{5}}{2}$.