Probabilistic techniques - tutorials

Problem set 4 – Problem set #4 - Lovász local lemma and Chernoff bound

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1. Prove that for every $0 < p < 1$ it holds that

$$\lim_{n\to\infty} \Pr[G(n,p) \text{ is } \frac{np^2}{2}\text{-vertex connected}] = 1.$$

[2]

Hint: For each pair of vertices u, v consider random variable $X_{u,v}$ = number of common neighbors of u and v, use Chernoff to bound $X_{u,v}$.

2. The van der Waerden number $W(r,k)$ is the smallest number n such that for every coloring of integers $\{1,\dots,n\}$ using r colors there exists an arithmetic progression of length k that is monochromatic. Prove that $W(2,k) \in \Omega(2^k/k)$. **[3]**

Hint: Let $c > 0$ and $n = c\frac{2^k}{k}$, we want to show that there exists a 2-coloring for which there is no monochromatic arithmetic progression of length k. Take α a random 2-coloring, i.e. color each vertex with probability 1/2. For P an arithmetic progression of length k set A_p to be the event *"P is monochromatic"*. Apply L.L.L. Use that every arithmetic progression of length k is determined by its starting point and the difference between two consecutive points.

3. Let $G = (V,E)$ be an undirected graph where each vertex $v \in V$ is assigned a list $L(v)$ of allowed colors. A proper list coloring of G assigns to each vertex $v \in V$ a color from $L(v)$ such that adjacent vertices have different colors. Let $k \geq 1$ be an integer and suppose the following conditions hold,

   - For each $v \in V$ we have that $|L(v)| \geq 8k$ and
   - For each $v \in V$ and $c \in L(V)$, there are at most k neighbors u of v such that $L(u)$ contains c.

   Show that G has a proper list coloring. **[4]**

Hint: For each edge (u,v) and color $c \in L(u) \cap L(v)$ consider the event $A_{u,v,c}$ = *"u and v have color c"*. Apply L.L.L.

4. The company Škoda has n^2 employees and it is time to select a new executive board composed of no more than n employees. The election goes as follows, each employee chooses an integer between 1 and n, without knowing the other employees choice. Then, the employees with the least frequent number are elected - if there are more least frequent numbers, we pick one of them randomly.

   Unfortunately, among the n^2 employees there are $n^2/10$ bad employees who want to ruin the company, the rest are good employees. The good employees choose the number during the election uniformly at random, while the bad employees can agree on a strategy and cooperate. Show that it is unlikely that more than one fifth of the board is composed of bad employees, i.e the probability of this happening goes to zero as n goes to infinity. **[4]**

Hint: If a large portion of bad employees got elected, then the random choice of good employees should differe from the expected number. Concretely, consider X_i to be the number of good employees that selected the number i and use Chernoff bound to control its value.

5. Consider $G(n,p)$ and let T_v be the number of triangles containing the vertex v. Prove that for every $0 < \epsilon < 1$ holds that

$$\lim_{n\to\infty} \Pr[\text{ for all } v \in [n]:\ (1-\epsilon)n^2p^3/2 \le T_v \le (1+\epsilon)n^2p^3/2] = 1$$

. **[4]**

Hint: For each $w \in V \setminus v$ consider the random variable $T_{v,w} =$ *"# triangles containing the edge v,w"*. Then $T_{v,w} = \sum_{u\neq v,w} T_{v,w,u}$ where $T_{v,w,u}$ is the indicator of the triangle v,w,u. Note that

$$T_{v,w} = \sum_{u\neq v,w} T_{v,w,u} = \sum_{u\neq v,w} I_{v,w}I_{u,v}I_{u,w} = I_{v,w}\sum_{u\neq v,w} I_{u,v}I_{u,w}$$

, where $I_{u,v}$ is the indicator of the edge, and this last sum is a sum of independent random variables, use Chernoff to control this sum.

6. Prove that for every $\epsilon > 0$, there exists $l_0 = l_0(\epsilon)$ and an infinite sequence of bits $a_1, a_2, \ldots,$ with $a_i \in \{0,1\}$ such that for every $l > l_0$ and every $i \ge 1$ the two binary vectors $u = (a_i, a_{i+1}, \ldots, a_{i+l-1})$ and $v = (a_{i+l}, \ldots, a_{i+2l-1})$ differ in at least $(\frac{1}{2} - \epsilon)l$ coordinates. **[3]**

Hint: By compactness, it is enough to prove for every lenght m the existence of a finite sequence of length m with the desired property. Consider a random sequence of that length and apply LLL.