Probabilistic techniques - tutorials

Problem set 3 – Problem set #3 - Markov and Chebyshev inequalities

Release: **November 9, 2023**. Hints: **November 23, 2023**. Deadline: **November 30, 2023**. Send solutions to honst+pt23@iuuk.mff.cuni.cz.

1. Let $n \geq 2$ be a positive integer and $v_1 = (x_1, y_1), \ldots, v_n = (x_n, y_n)$ vectors with x_i and y_i integers such that $x_i^2, y_i^2 \leq \frac{1}{10000}\frac{2^n}{n}$. Prove that there exist two non-empty disjoint subsets $I, J \subset [n]$ such that
$$\sum_{i \in I} v_i = \sum_{j \in J} v_j.$$
[3]

2. Let X denote the number of isolated vertices in $G(n, p(n))$ with $p(n) = c\frac{\ln(n)}{n}$. Show that
   (a) $\lim_{n\to\infty} \Pr[X = 0] = 1$ for $c > 1$. **[1]**
   (b) $\lim_{n\to\infty} \Pr[X \geq 1] = 1$ for $0 \leq c < 1$. **[3]**

3. Let X be a non-negative integer random variable such that $\mathbb{E}[X^2]$ is finite and nonzero. Prove that
$$\Pr[X = 0] \leq \frac{\mathrm{Var}[X]}{\mathbb{E}[X^2]}.$$
[2]

4. Prove that
$$\lim_{n\to\infty} \Pr[G(n, 1/2) \text{ has an induced cycle of lenght} > 3\log_2 n] = 0.$$
Remember that $(v_1, \ldots, v_k)$ is an induced cycle of length k if it satisfies that $v_i v_j$ is an edge if and only if $j = i + 1 \mod (k)$. **[3]**

5. Let X be a real random variable with $\mathrm{Var}[X] = \sigma^2$ and $\mathbb{E}[X] = 0$. For every real number $\lambda > 0$ prove the inequality
$$\Pr[X \geq \lambda] \leq \frac{\sigma^2}{\sigma^2 + \lambda^2}.$$
[2]

6. Show that there is a positive constant c such that the following holds: For any n vectors $a_1, \ldots, a_n \in \mathbb{R}^2$ satisfying $\sum_{i=1}^n \|a_i\|^2 = 1$ and $\|a_i\| \leq 1/10$, where $\|\cdot\|$ denotes the usual Euclidean norm. If $(\epsilon_1, ..., \epsilon_n)$ is a $\{-1, +1\}$-random vector obtained by choosing each ϵ_i randomly and independently with uniform distribution, then
$$\Pr\left[\left\|\sum_{i=1}^n \epsilon_i a_i\right\| \leq 1/3\right] \geq c.$$
[3]

7. Prove that for every set X of at least $4k^2$ distinct residue classes modulo a prime p, there is an integer a such that the set $\{ax \mod p \colon x \in X\}$ intersects every interval in $\{0, 1, \ldots, p-1\}$ of length at least p/k. **[3]**

Information about tutorials `https://kam.mff.cuni.cz/~dbulavka/teaching/ws2324/pt.html`