Probabilistic techniques - tutorials

Problem set 2 – Expectation and the method of alteration

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Below we denote by $R(k,l)$ the smallest number of vertices of a complete graph such that for every 2-coloring of its edges one can find either a K_k in the first color or a K_l in the second color.

1. Let M be an $n \times n$ matrix with entries uniformly independetly chosen from $\{-1,1\}$. Determine $\mathbb{E}[\det(M^2)]$. **[2]**

   Hint: Use that $\det(M) = \sum_{\sigma \in S_n} (-1)^{\mathrm{sgn}(\sigma)} X_{1,\sigma(1)} \cdots X_{n,\sigma(n)}$.

2. Let $G = (V, E)$ be a bipartite graph with n vertices such that each vertex $v \in V$ is assigned a list $L(v)$ of $\lceil log_2(n) \rceil + 1$ colors. Prove that there is a proper coloring of G such that each vertex v get a color from $L(v)$. **[2]**

   Hint: Define $L = \bigcup_{v \in V(G)} L(v)$ and partition $L = L_1 \cup L_2$ randomly. Assign each vertex of G a number $b(v) \in \{1,2\}$ depending on which part of the graph it is in. What is the expected number of vertices v such that $L(v) \cap L_{b(v)} = \emptyset$?

3. Let G be a graph on n vertices, let d be its average degree, Δ its maximal degree and $\alpha(G)$ the size of the largest independent set, i.e. induced subgraph with no edges. Show

   (a) $\alpha(G) \geq \frac{n}{\Delta+1}$. **[1]**

   Hint: Build the independent set greedily.

   (b) $\alpha(G) \geq \sum_{v \in V} \frac{1}{\deg(v)+1}$. **[3]**

   Hint: Pick a random permutation of vertices and repeat the greedy procedure from (a) with this random permutation. What is the expected size of the independent set which you will create?

   (c) $\alpha(G) \geq \frac{n}{d+1}$. **[1]**

   Hint: Use (b) and the harmonic arithmetic mean inequality.

4. Prove that there is a positive constant c such that every set B of n non-zero real numbers contains a subset A of size $|A| \geq cn$, such that there are no $b_1, b_2, b_3, b_4 \in A$ satisfying $b_1 + 2b_2 = 2b_3 + 2b_4$. **[2]**

   Hint: Consider c large enough and build a sub interval of $[0, c)$ that is free for that operation. Randomize the elements of B by multiplying by a random element of the interval $[0, c)$ and the desired elements are the ones landing in the sub-interval.

5. Suppose $p > n > 10m^2$, with p prime and let $0 < a_1 < \cdots < a_m < p$ be integers. Prove that there exists an integer x, $0 < x < p$ for which the m numbers

   $$(xa_i \mod p) \mod n$$

   are pairwise distinct. Notice that first we take modulo p, and then we take modulo n. **[3]**

   Hint: Consider X to be the random variable counting the number of collisions, let $X_{i,j}$ be the random variable that evaluates to 1 if xa_i is congruent to xa_j. Estime the probability that $X_{i,j} = 1$ using that there are roughly $\binom{p/n}{2}$ pairs of values that evaluate to the same value modulo n.

6. Prove that every 3-uniform hypergraph with n vertices and $m \geq n/3$ edges contains an independent set of size at least $\frac{2n^{3/2}}{3\sqrt{3}\sqrt{m}}$. **[2]**

   Hint: Select a set of vertices an random with probability p to be determined. Estive the number of edges there are in the induced subgraph $G[S]$ and remove one vertex from each edge.

7. Prove that $R(4,t) \in \Omega((t/\log(t))^2)$. You might want to first prove that for every $n, k, l \in \mathbb{N}$ and every $p \in [0,1]$ it holds that

$$R(k,l) > n - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{l} (1-p)^{\binom{l}{2}}.$$

[2]

   Hint: Pick a random graph on n vertices with edges with probability p. In expectation, how many cliques of size k does it have? How many independent sets of size l does it have? Throw away a vertex from each of those and prove the hint given in the problem statement this way. Now do some estimates and pick good values for k, l, n, and p.

8. Prove that for every n there is a bipartite graph with both parts of size n, at least $\Omega(n^{4/3})$ edges, but with no $K_{2,2}$ as a subgraph. **[2]**

   Hint: Pick a random bipartite graph, each edge with probability p. What is the expected number of edges? What is the expected number of $K_{2,2}$'s? Throw an edge from each $K_{2,2}$ away, you get a $K_{2,2}$-free graph. Now maximize the expected number of its edges with respect to p.