Probabilistic techniques - tutorials

Problem set 2 – Expectation and the method of alteration

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Below we denote by $R(k,l)$ the smallest number of vertices of a complete graph such that for every 2-coloring of its edges one can find either a K_k in the first color or a K_l in the second color.

1. Let M be an $n \times n$ matrix with entries uniformly independetly chosen from $\{-1,1\}$. Determine $\mathbb{E}[\det(M^2)]$. **[2]**

2. Let $G = (V,E)$ be a bipartite graph with n vertices such that each vertex $v \in V$ is assigned a list $L(v)$ of $\lceil log_2(n) \rceil + 1$ colors. Prove that there is a proper coloring of G such that each vertex v get a color from $L(v)$. **[2]**

3. Let G be a graph on n vertices, let d be its average degree, Δ its maximal degree and $\alpha(G)$ the size of the largest independent set, i.e. induced subgraph with no edges. Show

   (a) $\alpha(G) \geq \frac{n}{\Delta+1}$. **[1]**

   (b) $\alpha(G) \geq \sum_{v \in V} \frac{1}{\deg(v)+1}$. **[3]**

   (c) $\alpha(G) \geq \frac{n}{d+1}$. **[1]**

4. Prove that there is a positive constant c such that every set B of n non-zero real numbers contains a subset A of size $|A| \geq cn$, such that there are no $b_1, b_2, b_3, b_4 \in A$ satisfying $b_1 + 2b_2 = 2b_3 + 2b_4$. **[2]**

5. Suppose $p > n > 10m^2$, with p prime and let $0 < a_1 < \cdots < a_m < p$ be integers. Prove that there exists an integer x, $0 < x < p$ for which the m numbers
$$(xa_i \mod p) \mod n$$
   are pairwise distinct. Notice that first we take modulo p, and then we take modulo n. **[3]**

6. Prove that every 3-uniform hypergraph with n vertices and $m \geq n/3$ edges contains an independent set of size at least $\frac{2n^{3/2}}{3\sqrt{3}\sqrt{m}}$. **[2]**

7. Prove that $R(4,t) \in \Omega((t/\log(t))^2)$. You might want to first prove that for every $n, k, l \in \mathbb{N}$ and every $p \in [0,1]$ it holds that
$$R(k,l) > n - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{l} (1-p)^{\binom{l}{2}}.$$
   [2]

8. Prove that for every n there is a bipartite graph with both parts of size n, at least $\Omega(n^{4/3})$ edges, but with no $K_{2,2}$ as a subgraph. **[2]**

Information about tutorials `https://kam.mff.cuni.cz/~dbulavka/teaching/ws2324/pt.html`