Probabilistic techniques - tutorials

Problem set 1 – Basics

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By classical probability space we denote the probability space $(\Omega, 2^{\Omega}, \Pr)$ where Ω is a finite set and $\Pr[A] = |A|/|\Omega|$. We define $[n] = \{1, \ldots, n\}$. An n-uniform hypergraph H is a tuple (V, E) where the elements of E are subset of V of size n.

1. Consider a classical probability space on p elements with p a prime number. Let A and B be two events. Show that A and B are independent if and only if one of them is $\emptyset$ or Ω. **[1]**

   Hint: Use that for a prime number p and non-zero integers a and b, if $p|ab$ then $p|a$ or $p|b$.

2. Let A and B be disjoint subsets of $[n]$. Compute the probability that in a random permutation of $[n]$ the elements of A are in one cycle. Compute the probability that the elements of A are in one cycle and the elements of B are in one cycle, this two cycles do not need to be the same. Are the events "The elements of A are in one cycle" and "The elements of B are in one cycle" independent? **[4]**

   Hint: In both cases you can count the number of cycles. For the first part take into account the following identity known as "Hockey stick identity"$\sum_{i=a}^{n} \binom{i}{a} = \binom{n+1}{a+1}$. For the second part, let $|A| = a$, $|B| = b$, then the number of cycles is given by

   $$\sum_{l=0}^{n-a-b} \binom{n-a-b}{l} (a+l-1)! \sum_{k=0}^{n-a-b-l} \binom{n-a-b-l}{k} (b+k-1)!(n-a-b-l-k)!.$$

   Here you have to manipulate this sum until you get something nice just in terms of a and b.

3. Prove that there exists an absolute constant $c > 0$ such that for every natural number n and every $n \times n$ matrix A with pairwise distinct entries, there is a permutation of the columns of A such that no row contains an increasing subsequence of length greater than $c\sqrt{n}$. **[4]**

   Hint: Consider the event B_i ="row i has and increasing subsequence of length k"and $B = \cup_{i=1}^{n} B_i$. Now you have to upper bound $P[B_i]$ and use union bound to find c such that $P[B] < 1$.

4. Recall that $G(n, p)$ is a random graph on n vertices such that every pair of vertices form an edge with probability p independently of every other pair. Show that

   $$\lim_{n\to\infty} \Pr[G(n, 1/2) \text{ is connected}] = 1.$$

   [4]

   Hint: Let $A_{x,y}$ be the event "x and y have neither an edge nor a common neighbor". The event "G is disconnected"is contained in the union $\cup_{x,y} A_{x,y}$. Use union bound to show that in the limit $P[\cup_{x,y} A_{x,y}]$ is zero.

5. Prove that there exist constants $c_1, c_2 > 0$ such that for every integer n and m:

(a) If $m \geq c_1 n^2$, then a random mapping $[n] \to [m]$ is injective with probability at least 0.99. **[1]**

Hint: Use union bound over the events $A_y =$ "$f(x) = f(x') = y$".

(b) If $m \leq c_2 n^2$, then a random mapping $[n] \to [m]$ is injective with probability at most 0.01. **[2]**

Hint: Proceed by counting the number of injective function and use the AM-GM inequality.

6. Consider the classical probability space with 8 elements. Find an example of four events A, B, C and D such that all triples are independent, but the four events are not independent. **[2]**

   Hint: Try using subsets of size 4.

7. Let n be a positive integer such that $n \geq 4$. Let H be an n-uniform hypergraph with $|E(H)| \leq \frac{4^{n-1}}{3^n}$. Prove that there is a coloring of the vertices of H by 4 colors such that in every edge all 4 colors are present. The coloring doesn't need to be proper. **[2]**

   Hint: Consider the event $A_e =$"e is missing some color"and show that $P[A_e] < \frac{3^n}{4^{n-1}}$.

Information about tutorials `https://kam.mff.cuni.cz/~dbulavka/teaching/ws2324/pt.html`