Probabilistic techniques - tutorials

Classwork 4 – The second moment

1. Let $\omega(G)$ denote the size of the largest clique. The threshold function for the property $\omega(G(n,p)) \geq 4$ is $n^{-2/3}$.

2. Let $n \geq 3$ be a positive integer and $x_1, \dots, x_n$ with $x_i^2 \leq \frac{1}{10000}\frac{2^n}{n}$ for each $i \in [n]$. Prove that there exist two non-empty disjoint subsets $I, J \subset [n]$ such that
$$\sum_{i \in I} x_i = \sum_{j \in J} x_j.$$

3. Show that there is a positive constant c such that the following holds: For any n real numbers $a_1, \dots, a_n$ satifying $\sum_{i=1}^n a_i^2 = 1$, if $(\epsilon_1, \dots, \epsilon_n)$ is a $\{-1, +1\}$-random vector obtained by choosing each ϵ_i randomly and independently with uniform distribution to be either -1 or $+1$, then
$$\Pr\left[\left|\sum_{i=1}^n \epsilon_i a_i\right| \leq 1\right] \geq c.$$

4. A set of positive integers $\{x_1, \dots, x_n\}$ is said to have distinct sums if all sums
$$\sum_{i \in S} x_i, S \subseteq \{1, \dots, n\}$$
are distinct. Let $f(n) =$ maximal k such that there exist a set
$$\{x_1, \dots, x_k\} \subseteq \{1, \dots, n\}$$
with distinct sums. Show that
$$\lfloor \log_2(n) \rfloor \leq f(n) \leq \log_2(n) + \frac{\log_2(\log_2 n)}{2} + \mathcal{O}(1).$$