Probabilistic techniques - tutorials

Problem set #5 - Markov chains.

Release: **December 19, 2022**. Hints: **January 16, 2023**. Deadline: **January 23, 2023**. Send solutions to dbulavka+pt@kam.mff.cuni.cz.

1. By homogeneous discrete time Markov chain we mean a Markov chain $X_0, X_1, \dots$ such that for every n we have $\Pr[X_{n+1} = x | X_n = y] = \Pr[X_n = x | X_{n-1} = y]$.
   (a) Let $Y_0, Y_1, \dots$ be integers chosen independently uniformly from $\{-1, 0, 1\}$ and let $X_n = \max\{Y_0, \dots, Y_n\}$. Decide and justify whether $X_0, X_1, \dots$ is a homogeneous discrete time Markov chain. **[1]**
   (b) Let $Y_0, Y_1, \dots$ be integers chosen independently uniformly from $\{-1, 0, 1\}$ and let $X_n = \max\{Y_n, Y_{n-1}\}$. Decide and justify whether $X_0, X_1, \dots$ is a homogeneous discrete time Markov chain. **[1]**
2. Construct satisfiable 2-SAT formulae with n variables (for arbitrary large n) such that the randomized 2-SAT algorithm from the lecture takes $\Omega(n^2)$ steps (in expectation) to find a satisfying assignment. (Recall that in each step the algorithm picks an arbitrary unsatisfied clause and flips one of its variables) **[2]**
3. We say that a graph G is triangle 2-colorable if there is a 2-coloring of the vertices such that no triangle in G is monochromatic. (For example, a 3-colorable graph is triangle 2-colorable.) Let G be a triangle 2-colorable graph. Start with $c\colon V(G) \to \{0, 1\}$ being constant 0, and then do the following: while there is a monochromatic triangle in G (with respect to c), pick one of its vertices at random and flip its color. Show that the expected number of steps for this process to find a suitable triangle 2-coloring is polynomial in $|V(G)|$. **[3]**
4. Show that in one communicating class of a Markov chain, there cannot be a state which is recurrent and another state which is transient. **[2]**
5. In a finite Markov chain show that the following holds:
   (a) There is at least one recurrent state. **[2]**
   (b) All recurrent states are positive recurrent (i. e., the expected time until the process returns to this state is less than infinity). **[2]**
6. Let X be a Poisson random variable with mean $\lambda \in \mathbb{N}$. Prove that $\Pr[X \geq \lambda] \geq 1/2$. **[2]**
7. Throw m balls uniformly independently into n bins, where the bins are numbered from 0 to $n-1$. We say that there is a k-gap starting at j if bins $j, j+1, \dots, j+k-1$ are all empty.
   (a) Compute the expected number of k-gaps. **[1]**
   (b) Prove a Chernoff-like bound for the number of k-gaps X, that is, the inequality
   $$\Pr[X \geq (1+\delta)\,\mathbb{E}[X]] \leq e^{-c\,\mathbb{E}[X]\delta^2}$$
   where $0 < \delta < 1$ and c is a positive constant. **[4]**

Information about tutorials `https://kam.mff.cuni.cz/~dbulavka/teaching/ws2223/pt.html`