Probabilistic techniques - tutorials

Problem set 1 – Basics

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By classical probability space we denote the probability space $(\Omega, 2^\Omega, \Pr)$ where Ω is a finite set and $\Pr[A] = |A|/|\Omega|$. We define $[n] = \{1, \ldots, n\}$. An n-uniform hypergraph H is a tuple (V, E) where the elements of E are subset of V of size n.

1. Consider a classical probability space on p elements with p a prime number. Let A and B be two events. Show that A and B are independent if and only if one of them is $\emptyset$ or Ω. **[1]**

2. Let A and B be disjoint subsets of $[n]$. Compute the probability that in a random permutation of $[n]$ the elements of A are in one cycle. Compute the probability that the element of A are in one cycle and the element of B are in one cycle, this two cycles do not need to be the same. Are the events "The elements of A are in one cycle"and "The elements of B are in one cycle"independent? **[4]**

3. Prove that there exists an absolute constant $c > 0$ such that for every natural number n and every $n \times n$ matrix A with pairwise distinct entries, there is a permutation of the columns of A such that no row contains an increasing subsequence of length greater than $c\sqrt{n}$. **[4]**

4. Recall that $G(n, p)$ is a random graph on n vertices such that every pair of vertices form an edge with probability p independently of every other pair. Show that
$$\lim_{n\to\infty} \Pr[G(n, 1/2) \text{ is connected}] = 1.$$
[4]

5. Prove that there exist constants $c_1, c_2 > 0$ such that for every integer n and m:
   (a) If $m \geq c_1 n^2$, then a random mapping $[n] \to [m]$ is injective with probability at least 0.99. **[1]**
   (b) If $m \leq c_2 n^2$, then a random mapping $[n] \to [m]$ is injective with probability at most 0.01. **[2]**

6. Consider the classical probability space with 8 elements. Find an example of four events A, B, C and D such that all triples are independent, but the four events are not independent. **[2]**

7. Let n be a positive integer such that $n \geq 4$. Let H be an n-uniform hypergraph with $|E(H)| \leq \frac{4^{n-1}}{3^n}$. Prove that there is a coloring of the vertices of H by 4 colors such that in every edge all 4 colors are present. The coloring doesn't need to be proper. **[2]**

Information about tutorials `https://kam.mff.cuni.cz/~dbulavka/teaching/ws2223/pt.html`