Mathematics++

Problem set 2 – Fourier analysis

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Theorem (Riesz-Thorin interpolation). Let $p_0, q_0, p_1, q_1 \in [1, \infty]$, $(\frac{1}{0} = \infty)$. Let $c_0, c_1 > 0$ be real numbers. Let p_t, q_t that satisfy

$$\frac{1}{p_t} = \frac{t}{p_1} + \frac{1-t}{p_0}, \qquad \frac{1}{q_t} = \frac{t}{q_1} + \frac{1-t}{q_0}$$

for each $t \in [0, 1]$. Next, let $c_t = c_0^{1-t} c_1^t$. Let $T\colon \mathbb{C}^X \to \mathbb{C}^Y$ be linear and X, Y finite sets. Let's assume it satisfies $\|Tf\|_{L_{q_t}} \leq c_t \|f\|_{L_{p_t}}$ for $t = 0$ and $t = 1$. Then the same is true for all $t \in [0, 1]$.

1. Show that the characters are eigenvectors for the convolution operator, with any function. That is for a function f and character χ we have that $f * \chi = \lambda \cdot \chi$. Compute the eigenvalue λ of χ. **[2]**

2. Let G be an abelian group and H a subgroup of G. Let $f\colon G \to \mathbb{C}$ be a function and $a \in G$. Show that
$$\frac{1}{|H|} \sum_{x \in H} f(x + a) = \sum_{y \in H^\perp} \widehat{f}(y) \chi_y(a),$$
where $H^\perp = \{a \in G\colon \chi_a(x) = 1, \forall x \in H\}$. **[4]**

3. A function $f\colon \{-1, 1\}^n \to \{-1, 1\}$ is monotone if we have that $f(x) \leq f(y)$ whenever $x_k \leq y_k$ is true for each k.
   (a) Show that for any monotone function $f\colon \{-1, 1\}^n \to \{-1, 1\}$ we have that $\mathrm{Inf}_i(f) = \hat{f}(\{i\})$. **[3]**
   (b) Show that for n odd the function $f(x) = \mathrm{sgn}(\sum_i x_i)$ maximizes the total influence among monotone functions of n variables from $\{-1, 1\}^n$ to $\{-1.1\}$. By total influence we mean $\mathrm{Inf}(f) = \sum_{i=1}^n \mathrm{Inf}_i(f)$. **[3]**

4. For p prime and $r \in \mathbb{Z}_p$, we define $\mathrm{Gau}(r) := \sum_{x \in \mathbb{Z}_p} e(rx^2/p)$ (the so-called Gaussian sum). Prove that
   (a) $\mathrm{Gau}(rs^2) = \mathrm{Gau}(r)$ for $s \in \mathbb{Z}_p \setminus \{0\}$, **[2]**
   (b) if -1 is not a quadratic residue in $\mathbb{Z}_p$, then $\mathrm{Gau}(-r) = -\mathrm{Gau}(r)$, **[4]**
   (c) $\mathrm{Gau}(1)^2 = \pm p$ for p prime different from 2. **[4]**

5. Let $p, q \geq 1$ and let $1/p + 1/q = 1$. Then for a finite group G and mapping $f \in \mathbb{C}^G$, prove
   (a) $\|f\|_p \geq \hat{\|}\widehat{f}\hat{\|}_q$ for $p \in [1, 2]$, **[2]**
   (b) $\|f\|_p \leq \hat{\|}\widehat{f}\hat{\|}_q$ for $p \in [2, \infty]$. **[2]**
   (c) Further, let $p, q, r \in [1, \infty]$ be such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. Prove that for any mappings $f, g \in \mathbb{C}^G$, $\|f * g\|_r \leq \|f\|_p \|g\|_q$. **[4]**