Mathematics++

Problem set 1 – Harmonic analysis

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Send solutions to dbulavka+mpp@kam.mff.cuni.cz.

The box product $G\square H$ of graphs G and H is a graph whose vertex set is the cartesian product $V(G) \times V(H)$, and a pair of vertices $(u, u'), (v, v') \in V(G) \times V(H)$ is an edge in $G\square H$ if either $u = v$ and $u'v' \in E(H)$, or $u' = v'$ and $uv \in E(G)$. The box product is associative.

1. Let G be a finite abelian group and let $S \subseteq G$ be a set such that $0 \notin S$ and S is symmetric (i.e., $S = -S$). The Cayley graph $\mathrm{Cay}(G; S)$ is the graph (G, E), where $ab \in E$ whenever $b - a \in S$. Let χ be a character of G, and let $A = (a_{ij})$ be the adjacency matrix of $\mathrm{Cay}(G; S)$, i.e., $a_{ij} = 1$ if ij is an edge and $a_{ij} = 0$ otherwise.

   (a) Consider a vector $x \in \mathbb{C}^G$ such that for $a \in G$ we have $x_a = \chi(a)$. Prove that x is an eigenvector of $\mathrm{Cay}(G; S)$ (i.e., of the matrix A).

   Hint: Let $Ax = y$, then $y_i = \sum_{j\in G} a_{i,j}\chi(j)$, add and substract i in $\chi(j)$ and use the fact that χ is a group homomorphism and S is symmetric. **[2]**

   (b) For $i = 1, \ldots, d$, let G_i be a group and $S_i \subseteq G_i$ a symmetric subsets such that $0 \notin S_i$ and let $G = \prod_{i=1}^{d} G_i$. Let χ_i be a character of G_i and set $x \in \mathbb{C}^G$ as $x_{(a_1,\ldots,a_d)} = \chi_1(a_1)\cdots\chi_d(a_d)$. Show that x is an eigenvector of $\square_{i=1}^{d} \mathrm{Cay}(G_i; S_i)$ and compute its eigenvalue.

   Hint: For each $i \in [d]$ and each $s \in S_i$ consider the vector $v_s \in G$ given by $(v_s)_j = s$ is $j = i$ and 0 otherwise and let S be the set of all such v_s. Show that the Cayley graph $\mathrm{Cay}(G, S)$ coincides with $\square_{i=1}^{d} \mathrm{Cay}(G_i; S_i)$. Then use that $\hat{G}$ is isomorphic to $\prod_{i=1}^{d} \hat{G}_i$ as shown in the tutorial. **[2]**

   (c) For $n_1, \ldots, n_d$ positive integers, find all the eigenvalues of the graph $\square_i^d C_{n_i}$, where C_n is the cycle with n vertices.

   Hint: $C_n = \mathrm{Cay}(\mathbb{Z}/n\mathbb{Z}, \{-1, 1\})$, $e^{2\pi i a/n} + e^{-2\pi i a/n}$ and apply the above item. To see that it is all use theorem about Fourier basis viewed in lecture. **[3]**

   (d) Compute all eigenvalues of Q_d, the d-dimensional hypercube: $V(Q_d) = \{0, 1\}^d$ and ab is an edge whenever a and b differ in exactly one coordinate.

   Hint: Consider $Q_d = \mathrm{Cay}(\mathbb{Z}_2^n; \{e_i : i \in [n]\})$ and use the first item with the characters $\chi_a = (-1)^{\sum_{i\in[n]}}$. To argue that these are all use theorem viewd in lecture. **[3]**

2. Let $f \colon \{0, 1\}^n \to \{0, 1\}$ denote a function. The influence of the k-th variable on f is defined by

$$\mathrm{Inf}_k(f) = \Pr[x \in \mathbb{Z}_2^n : f(x) \neq f(x + e_k)].$$

   (a) Determine the influence of the majority function: for an odd n the function $\mathrm{Maj}(x_1, \ldots x_n) : \{0, 1\}^n \to \{0, 1\}$ is defined as the more frequent value among $x_1, \ldots, x_n$.

   Hint: How do the vectors in $\mathbb{Z}_2^n$ look like for which the i-th variable can change the outcome of Maj? **[3]**

(b) Using a formula in disjunctive normal form, construct an example of $f\colon \mathbb{Z}_2^n \to \{0,1\}$ with $\mathrm{Inf}_k(f) = \frac{2\ln(n)}{n}(1+o(1))$ for every k.

Hint: Use the function $f_{b,c}\colon \{0,1\}^{bc} \to \{0,1\}$ defined by

$$x \longmapsto \bigvee_{i=1}^{c} \bigwedge_{j=1}^{b} x_{i,j}$$

and think about how to treat non-integer b and c. **[4]**

3. Find the matrix of the linear mapping given by the Fourier transform on $\mathbb{Z}_n$. Explicitly, find a matrix M_n such that for every $f\colon \mathbb{Z}_n \to \mathbb{C}$ we have

$$(\hat{f}(0), \ldots, \hat{f}(n-1))^t = M_n(f(0), \ldots, f(n-1))^t.$$

Compute $\det(M_n)$ and thus re-prove the fact that the Fourier transform is a bijection.

Hint: Consider the Vandermonde matrix. **[4]**

4. Let G be an abelian group, and let $f\colon G \to \mathbb{C}$ be a function that is not identically zero. We define the support of f to be the set $\mathrm{Supp}(f)$ of all $x \in G$ for which $f(x) \neq 0$. Prove that

(a) $\mathrm{Supp}(f * g) \subseteq \mathrm{Supp}(f) + \mathrm{Supp}(g)$.

Hint: Work out from the definition of what it means that $x \in \mathrm{Supp}(f * g)$, noting that if the sum is non-zero then some summation must also be non-zero. **[2]**

(b) $\|f * g\|_\infty \leq \|f\|_p \|g\|_q$, where $1/p + 1/q = 1$.

Hint: Use Hölder's inequality and be carefull with the definitions of the norm. **[2]**

(c) $\widehat{f \cdot g}(\chi) = \sum_{\psi \in G} \hat{f}(\chi - \psi)\hat{g}(\psi)$.

Hint: Use the inverse Fourier transform. **[2]**

(d) $|\,\mathrm{Supp}(f)| \cdot |\,\mathrm{Supp}(\hat{f})| \geq |G|$.

Hint: Notice that $\sum_{x \in G} |f(x)|^2 \leq |\,\mathrm{Supp}(f)| \max_{x \in G} |f(x)|$ and $\max_{x \in G} |f(x)| \leq \frac{1}{|G|}|\hat{f}(x)|$. Finally use Cauchy-Schwarz with $|\hat{f}(x)|$ and the constant funtion 1. **[3]**