Mathematics++

Problem set 1 – Harmonic analysis

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The box product $G\square H$ of graphs G and H is a graph whose vertex set is the cartesian product $V(G) \times V(H)$, and a pair of vertices $(u,u'), (v,v') \in V(G) \times V(H)$ is an edge in $G\square H$ if either $u = v$ and $u'v' \in E(H)$, or $u' = v'$ and $uv \in E(G)$. The box product is associative.

1. Let G be a finite abelian group and let $S \subseteq G$ be a set such that $0 \notin S$ and S is symmetric (i.e., $S = -S$). The Cayley graph $\mathrm{Cay}(G;S)$ is the graph (G,E), where $ab \in E$ whenever $b-a \in S$. Let χ be a character of G, and let $A = (a_{ij})$ be the adjacency matrix of $\mathrm{Cay}(G;S)$, i.e., $a_{ij} = 1$ if ij is an edge and $a_{ij} = 0$ otherwise.

   (a) Consider a vector $x \in \mathbb{C}^G$ such that for $a \in G$ we have $x_a = \chi(a)$. Prove that x is an eigenvector of $\mathrm{Cay}(G;S)$ (i.e., of the matrix A). **[2]**

   (b) For $i = 1, \ldots, d$, let G_i be a group and $S_i \subseteq G_i$ a symmetric subsets such that $0 \notin S_i$ and let $G = \prod_{i=1}^d G_i$. Let χ_i be a character of G_i and set $x \in \mathbb{C}^G$ as $x_{(a_1,\ldots,a_d)} = \chi_1(a_1)\cdots\chi_d(a_d)$. Show that x is an eigenvector of $\square_{i=1}^d \mathrm{Cay}(G_i;S_i)$ and compute its eigenvalue. **[2]**

   (c) For $n_1, \ldots, n_d$ positive integers, find all the eigenvalues of the graph $\square_i^d C_{n_i}$, where C_n is the cycle with n vertices. **[3]**

   (d) Compute all eigenvalues of Q_d, the d-dimensional hypercube: $V(Q_d) = \{0,1\}^d$ and ab is an edge whenever a and b differ in exactly one coordinate. **[3]**

2. Let $f\colon \{0,1\}^n \to \{0,1\}$ denote a function. The influence of the k-th variable on f is defined by

$$\mathrm{Inf}_k(f) = \Pr[x \in \mathbb{Z}_2^n\colon f(x) \neq f(x+e_k)].$$

   (a) Determine the influence of the majority function: for an odd n the function $\mathrm{Maj}(x_1,\ldots x_n) : \{0,1\}^n \to \{0,1\}$ is defined as the more frequent value among $x_1, \ldots, x_n$. **[3]**

   (b) Using a formula in disjunctive normal form, construct an example of $f\colon \mathbb{Z}_2^n \to \{0,1\}$ with $\mathrm{Inf}_k(f) = \frac{2\ln(n)}{n}(1+o(1))$ for every k. **[4]**

3. Find the matrix of the linear mapping given by the Fourier transform on $\mathbb{Z}_n$. Explicitly, find a matrix M_n such that for every $f\colon \mathbb{Z}_n \to \mathbb{C}$ we have

$$(\hat{f}(0), \ldots, \hat{f}(n-1))^t = M_n(f(0), \ldots, f(n-1))^t.$$

   Compute $\det(M_n)$ and thus re-prove the fact that the Fourier transform is a bijection. **[4]**

4. Let G be an abelian group, and let $f\colon G \to \mathbb{C}$ be a function that is not identically zero. We define the support of f to be the set $\mathrm{Supp}(f)$ of all $x \in G$ for which $f(x) \neq 0$. Prove that

   (a) $\mathrm{Supp}(f * g) \subseteq \mathrm{Supp}(f) + \mathrm{Supp}(g)$. **[2]**

   (b) $\|f * g\|_\infty \leq \|f\|_p \, \|g\|_q$, where $1/p + 1/q = 1$. **[2]**

   (c) $\widehat{f \cdot g}(\chi) = \sum_{\psi \in G} \hat{f}(\chi - \psi)\hat{g}(\psi)$. **[2]**

   (d) $|\,\mathrm{Supp}(f)| \cdot |\,\mathrm{Supp}(\hat{f})| \geq |G|$. **[3]**