Mathematics++

Classwork 2 – Fourier Analysis

March 7th, 2023.

1. For $S \subseteq \{1, \dots, n\}$ and $x \in \mathbb{Z}_2^n$ let

$$\chi_S(x) = (-1)^{\sum_{i \in S} x_i}.$$

For $a = (a_1, \dots, a_k), x = (x_1, \dots, x_k) \in \oplus_{i=1}^k \mathbb{Z}/n_i\mathbb{Z}$ let

$$\chi_a(x) = e(\sum_{i=1}^{k} \frac{a_i x_i}{n_i}).$$

Here $e(x) = e^{2\pi i x}$.

   (a) Prove that the mappings χ_S and χ_a given above are characteres.

   (b) Prove that the assignment $a \mapsto \chi_a$ is injective group homomorphism.

2. Compute the Fourier transform as well as the inverse Fourier transform for the following functions:

   (a) χ_a

   (b) δ_a

   (c) $\delta_{-1} + \delta_0 + \delta_1$ for $G = \mathbb{Z}_n$

   (d) $\sum_{i=0}^{k-1} \delta_{il}$ for $G = \mathbb{Z}_{kl}$

3. Verify that

   (a) $\hat{f}(0) = \mathbb{E}[f]$ for every finite abelian group and every function f.

   (b) $\left\|\hat{f}\right\|_\infty \leq \|f\|_1$.

4. Fix $p \in G$ and $c \in \mathbb{C}$. We define operators T_p and P_c to map a function $f \colon G \to \mathbb{C}$ to another function defined as follows: $(T_p f)(x) = f(x + p)$ and $(P_c f)(x) = cf(x)$. In case G is a field and $p \neq 0$, we also define $(S_p f)(x) = f(px)$. Prove that

   (a) $\hat{T_p f}(a) = \chi_a(p)\hat{f}(a)$

   (b) $\hat{P_c f}(a) = c\hat{f}(a)$

   (c) $\hat{S_p f}(a) = \hat{f}(a/p)$

5. Let $f, g \colon G \to \mathbb{C}$ two functions defined on a finite abelian group, their convolution is given by $(f * g)(z) = \frac{1}{|G|} \sum_{x \in G} f(x)g(z - x)$. Verify that for function $f, g, h \colon G \to \mathbb{C}$ defined on a finite abelian group and for complex numbers a, b the convolution satisfies the following:

   (a) $f * g = g * f$,

   (b) $f * (g * h) = (f * g) * h$,

(c) $(af + bg) * h = a(f * h) + b(g * h)$

6. Let G be a k-regular non-bipartite graph with n vertices and adjacency matrix A. Define $\mu = \max\{|\lambda| \colon \lambda \in Spec(A), |\lambda| \neq k\}$. Show that the diameter of G satisfies $d \leq \frac{\log(n-1)}{\log(k)-\log(\mu)} + 1$.

7. Let $1 \leq r < n$ and set $G = Cay(\mathbb{Z}/n\mathbb{Z}, B(r))$ where $B(r) = \{\pm 1, \ldots, \pm r \mod n\}$.

   (a) Show that G is regular.

   (b) Decide if G is Ramanujan, i.e., let G be k-regular then it is Ramanjuan if for $\lambda \in Spec(Adj(G))$ such that $|\lambda| \neq k$ then $|\lambda| \leq 2\sqrt{k-1}$.