Mathematics++

Classwork 1 – Harmonic analysis

February 21st, 2023.

Definition 1. Set $\mathbb{T} = \{z \in \mathbb{C}\colon \|z\| = 1\}$. Let $G = (G, +, 0)$ be a finite abelian group. We say that a function $\chi\colon G \to \mathbb{T}$ is a character of G if it is a group homomorphism. Let $\widehat{G}$ be the set of all characters of G.

A Hilbert space H is an n-dimensional vector space over $\mathbb{C}$ with inner product $\langle\cdot,\cdot\rangle\colon H \times H \to \mathbb{C}$. If $v_1, \dots, v_n$ is an orthogonal basis then for $f \in H$ we have that $f = \sum_{i=1}^{n} \frac{\langle v_i, f\rangle}{\langle v_i, v_i\rangle} v_i$.

Let $G = (G, *, 1)$ and $H = (H, *', 1')$ be two groups, a function $f\colon G \to H$ is a group homomorphism is $f(1) = 1'$ and $f(a * b) = f(a) *' f(b)$.

1. If $z \in \mathbb{T}$, then $z^{-1} = \bar{z}$.
2. If χ is a character of G, then $1/\chi$ is a character of G.
3. If $\chi\colon G \to \mathbb{C} \setminus \{0\}$ group homomorphism, then χ is character of G.
4. Show that $\widehat{\bigoplus_{i=1}^{d} G_i}$ and $\bigoplus_{i=1}^{d} \widehat{G_i}$ are isomorphic as groups.
5. Let $L_2(\mathbb{Z}/n\mathbb{Z}) = \{f\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}\}$ with inner product given by $\langle f, g\rangle = \sum_{i=1}^{n} f(i)\overline{g(i)}$.
   (a) Verify that it is interior product.
   (b) Show that $L_2(\mathbb{Z}/n\mathbb{Z})$ is an n-dimensional vector space over $\mathbb{C}$ with basis consisting of $\delta_0, \dots, \delta_{n-1}$ where $\delta_i(j) = 1$ if $i = j$ and 0 otherwise.
   (c) Show that δ_i's form an orthonormal basis of $L_2(\mathbb{Z}/n\mathbb{Z})$.
   (d) Since δ_i'a form orthonormal basis we can write any $f \in L_2(\mathbb{Z}/n\mathbb{Z})$ as $f(x) = \sum_{i=0}^{n-1} \delta_i(x) f(i)$. Show that $T\colon L_2(\mathbb{Z}/n\mathbb{Z}) \to \mathbb{C}^n$ given by $T(f) = (f(0), \dots, f(n-1))$ is a vector space isomorphism.
6. Let $f\colon \{0,1\}^n \to \{0,1\}$. The influence of the k-th variable on f is defined by

   $$\mathrm{Inf}_k(f) = \Pr[x \in \mathbb{Z}_2^n\colon f(x) \neq f(x + e_k)].$$

   Compute the k-th influence of the following functions:
   (a) $f(x) = x_1$.
   (b) $f(x) = \sum_{i=1}^{n} x_i \mod 2$.
7. Show that $\mathbb{T}$ and $\mathbb{R}/\mathbb{Z}$ isomorphic as groups.