Topological methods in combinatorics - tutorials

Problem set 5 – Tverberg's theorem, chessboard complexes, polyhedral complexes

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Submit solution to: skotnica at kam.mff.cuni.cz

Definition 1. A *(geometric) polyhedral complex* is a collection of polyhedra $\mathsf{M} = \{M_1, \ldots, M_k\}$ such that each M_i is a bounded polyhedron in $\mathbb{R}^d$ for some d and such that the following holds:

1. If $M \in \mathsf{M}$ and F is a face of M, then $F \in \mathsf{M}$.
2. If $M_1, M_2 \in \mathsf{M}$, then $M_1 \cap M_2$ is a face of both M_1 and M_2.

1. Prove that there are at least two Tverberg 3-partitions of every set X of seven points in the plane. In other words, the points from X can be divided in two different ways into three pairwise disjoint sets X_1, X_2, X_3 such that $\text{conv}(X_1) \cap \text{conv}(X_2) \cap \text{conv}(X_3) \neq \emptyset$. **[2]**

   Hint: One partition is guaranteed by Tverberg's theorem. Assuming this partition color the points and use Blagojević–Matschke–Ziegler theorem.

2. Let X be a set of 11 points in the plane, four of them are red, another four green and the rest (three) blue. Prove that there is a subset of X having Tverberg rainbow 3-partition. In other words, there exist pairwise disjoint sets $X_1, X_2, X_3 \subseteq X$ such that $\text{conv}(X_1) \cap \text{conv}(X_2) \cap \text{conv}(X_3) \neq \emptyset$ and no X_i contains two points of the same color. **[2]**

   Hint: Add a point/points to X and use Blagojević–Matschke–Ziegler theorem.

3. Let K be a simplicial complex defined as follows. Consider k chessboards of sizes $s_1 \times (s_1+1), s_2 \times (s_2+1), \ldots, s_k \times (s_k+1)$ (the number of columns is greater by one than the number of rows). Each vertex corresponds to a placement of one rook on any of the chessboards. Simplices correspond to placements of rooks such that no rook threatens any other; that is, no two rooks of the same chessboard share a row or a column.

   Prove that K is an orientable pseudomanifold. **[4]**

   Hint: Let N be the dimension of the complex. For each face F of dimension N assume a mapping $f_i^F : \{1, \ldots, s_i\} \to \{1, \ldots, s_i + 1\}$. Each mapping represents one of the chessboards and we set $f_i^F(j)$ to the number of column containing the rook of the j-th row on the i-th chessboard forming F. Use this mapping to define an orientation of each face of dimension N.

4. Let M is a polyhedral complex. Prove that there is a simplicial subdivision K of M without new vertices. In other words, prove, that there is a simplicial complex K such that $V(\mathsf{K}) = V(\mathsf{M})$ and for each simplex $\sigma \in \mathsf{K}$ there is $M \in \mathsf{M}$ such that $\sigma \subseteq M$.

   Hint: A subdivision K is called *conical* if for each $M \in \mathsf{M}$ there is a vertex $v(M) \in M$ such that $v(M)$ is contained in every maximal (relatively inside M) simplex of K subdividing M. Look for a conical subdivision. **[4]**

   Hint: Order the vertices of M and use induction on the dimension. During the inductive step for dimension k use a triangulation of $(k-1)$-skeleton guaranteed by the induction hypothesis.

Information about tutorials `https://kam.mff.cuni.cz/~dbulavka/teaching/ss2122/topomethods.html`