Topological methods in combinatorics - tutorials

Problem set 1 – Basics of general and algebraic topology

Submitted: **25.02.2022** - Hints: **11. 03. 2021** - Deadline: **18. 03. 2021**

Submit solution to: dbulavka+tm at kam.mff.cuni.cz

Please include with your solutions a statement if you want your score to be displayed on the website as well as a nickname.

1. Let X and Y be topological spaces, $f\colon X \to Y$ continuous function and $M, N \subset X$. Decide if the following claims hold, justify in each case.

   (a) If M is closed set, then $f(M)$ is closed. **[1]**

   Hint: Think of a subset of $\mathbb{R}$ that is not closed.

   (b) If M is open set, then $f(M)$ is open. **[1]**

   Hint: Think of a subset of $\mathbb{R}$ that is not open

   (c) If M is connected then $f(M)$ is connected. **[1]**

   Hint: Verify that it satisfies the definition of connectivity.

   (d) If M is disconnected, then $f(M)$ is disconnected. **[1]**

   Hint: Think of a non-injective function

   (e) If M is closed and N compact, then $M \cap N$ is compact. **[1]**

   Hint: Verify that it satisfies the definition of compactness.

2. Let $v_1, \ldots, v_n$ be the vertices of a convex polygon P in $\mathbb{R}^2$ with the origin in the interior. Show that ∂P and S^1 are homeomoprhic. **[3]**

   Hint: Consider the function $f\colon P \to S^1$ given by $f(x) = x/\|x\|$ and use the proposition showed in the tutorial about compact Hausdorff spaces.

3. Show that the topology on $\mathbb{R}^n$ given by the basis

   $$\{B(x,\epsilon)\colon x \in \mathbb{R}^n, \epsilon \in \mathbb{R} \text{ such that } \epsilon > 0\}$$

   coincides with the one given by $\epsilon - \delta$ definition. Here $B(x,\epsilon)$ is the open ball with center x and radius ϵ. **[2]**

   Hint: Compare the two definition of open set. Notice that any point x in an open set U in the $\epsilon - \delta$ topology has an open ball $B(x,\epsilon) \subset U$. Write U as the union of these.

4. Let $1 \leq r \leq n/2$, the Kneser graph $KG_{n,r}$ is the graph with vertex set all the r-subsets of $[n]$, i.e. $V(KG_{n,r}) = \{S \subset [n]\colon |S| = r\}$. A pair of r-sets S, T form an edge in $KG_{n,r}$ if $S \cap T = \emptyset$. Show that the chromatic number of $KG_{n,r}$ is at most $n - 2r + 2$. **[2]**

   Hint: Consider the coloring given by $c(F) = \min\{F, n - 2r + 2\}$.

5. (Gluing lemma for continuous functions) Let X be a topological space and $A_1, \ldots, A_n$ closed subspaces of X such that $X = \cup_{i=1}^n A_i$. Let $f\colon X \to Y$ function between topological spaces. Show that f is continuous if and only if the restriction of f to each A_i is continuous. **[2]**

   Hint: Verify that it satisfy the definition of continuous function with closed sets.

6. Show that the connected components of a topological space X are closed subsets. **[2]**

   Hint: Assume it is not the case. A connected component C is a maximal connected set. Consider the closure $Cl(C)$ of C. The $Cl(C)$ must be disconnected. Following this argument you will reach a contradiction.

7. Show that two connected graphs with the same number of vertices and edges are homotopically equivalent. **[4]**

   Hint: Let G be a graph, this is a simplicial complex so its geometrical realization is well defined. Let T be a tree, then its geometric realization, $|T|$, is contractible, i.e. homotopy equivalent to a point. In the case of a tree we can see this by showing homotopy equivalence to a vertex, by contracting one edge at a time, and arguing by transitivity. Once we have that we can see that a graph G is homotopy equivalent to a point with several S^1 attached to it by taking a spanning tree. How many S^1 are attached to the point?