Mathematics++

Practicals 5 – Functional analysis

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All the vector spaces (also called linear spaces) are over the field $\mathbb{R}$.

Definition: Let E be a normed linear space. A **closed hyperplane** is every set of the form $H = \{x \in E : f(x) = \alpha\}$ where $f \in E^*$, $f \neq 0$ and $\alpha \in \mathbb{R}$. (This is the same as translations of maximal proper subspaces).

Spaces and norms:

- $\mathcal{C}([0,1])$: continuous functions $[0,1] \to \mathbb{R}$ with norm $\|f\| = \max\{|f(t)| : t \in [0,1]\}$.
- c: convergent sequences with norm $\|x_n\| = \sup\{|x_n| : n \in \mathbb{N}\}$.
- c_0: sequences convergent to 0, subspace of c.
- l^∞: bounded sequences, same norm as in c.
- $\mathcal{L}^p$: measurable functions on X with norm $\|f\|_p = \left(\int |f|^p \, \mathrm{d}\mu\right)^{1/p}$. This is not a norm, functions that are zero almost everywhere have norm zero.
- L^p: $\mathcal{L}^p$ modulo functions that are zero almost everywhere.

1. Find a function $f\colon \mathbb{R} \to \mathbb{R}$ such that $|f(x) - f(y)| < |x - y|$ but f is not a contraction.

   Řešení: Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined on unit lenght intervals $[i-1, i]$ by sending it to the interval $[0, 1 - 1/2^i]$ for $i = 1, \ldots$. For i odd we have that $f(i-1+t) = (1-1/2^i)t$ and for i even we have $f(i-1+t) = (1-1/2^{i-1})(1-t)$. We can verify that at the end points it coincides and consequently it is well defined. We do the analogous symmetrically for negative number. It satisfies the hypothesis: (1) for x, y in the same interval we have that $|f(x)-f(y)| = (1-1/2^i)|t-t'| = (1-1/2^i)|x-y|$; (2) on consecutive intervals for i odd, $x = i-1+t$ and $y = (i+1)-1+t'$; we have that $|f(x)-f(y)| \leq |f(x)-f(i)|+|f(i)-f(y)| \leq |x-i|+|i-y| = |x-y|$;(3) on non-consecutive intervals is immediate since $|f(x)-f(y)| < 1$. The function f is not contraction since $|f(i-1) - f(i)| = 1 - 1/2^i$ which goes to 1.

2. Find a function $f : \mathbb{R} \to \mathbb{R}$ such that $|f(x) - f(y)| < |x - y|$ but f has no fixed point.

   Řešení: Let f be the function defined in the first exercise. Its only fixed point is the 0, it is enough to shift it by a little bit, say $g(x) = f(x) + 1/8$. This way in the interval $[0,1]$ the image is strictly inside $[0,1]$ and for the remaining number it is below 2.

3. Show that every subspace of a normed linear space of finite dimension is closed. Find a counterexample for a space of infinite dimension.

   Řešení: Because any two normed linear spaces with the same dimension are topologically isomorphic we can consider without loss of generality that our space is $\mathbb{R}^n$. A subspace of $\mathbb{R}^n$ is isomorphic to $\mathbb{R}^k$ for some k which is an intersection of hyperplanes and consequently closed. For the counterexample, consider in c_0 the subspace generated by elements e_i, i.e. $(e_i)_j = 1$ if $i = j$ and 0 otherwise. Consider the sequence a_i given by $(a_i)_j = 1/j$ for $j < i$ and 0 otherwise. This sequence converges to $(1/j)_{j\in\mathbb{N}}$ which is in c_0 but not in the subspace.

4. Show that complement of every proper subspace of a normed linear space is dense.

   Řešení: Let $W \subseteq V$ closed proper subspace and $U = V \setminus W \neq \emptyset$ because W is proper. Let B be open ball, we want to verify that $B \cap U \neq \emptyset$. Assume that it is not the case, i.e. $B \cap U = \emptyset$, then $B = B(x, r) \subseteq W$. Take $y \in U \neq \emptyset$, and take $\lambda = \frac{r}{2|x-y|}$, then $z = x + (y - x)\lambda \in B(x, r) \subseteq W$, and consequently $y = x + \frac{1}{\lambda}(z - x) \in W$, which is a contradiction.

5. Show that unit ball in a Hilbert space of inifinite dimension is not compact.

   Řešení: Consider an orthonormal basis. Every pair of elements have distance $\sqrt{2}$, i.e. no convergent subsequence.

6. Prove Mazur theorem: Let C be an open convex subset of a normed linear space E and $z \in E \setminus C$. Then there exists a closed hyperplane $H \subset E$ such that $z \in H$ and $H \cap C = \emptyset$.

   Řešení: Without loss of generality assume that $z = 0$. Let $G = \{\lambda c \colon c \in C, \lambda > 0\}$ be the cone over C. Define the function

$$p(x) \colon = \inf\{\|x + y\| : y \in G\}.$$

   Let g_0 such that $\|g_0\| = 1$ and set $f(\lambda g_0) = \lambda$ defined on the subspace $Y \colon = \{\lambda g_0 \colon \lambda \in \mathbb{R}\}$.

   We apply Hahn-Banach lemma: X linear space and $p \colon X \to \mathbb{R}$ continuous function such that $p(x + y) \leq p(x) + p(y)$ and $p(\lambda x) = \lambda p(x)$ for all $\lambda \geq 0$, $x, y \in X$; let Y subspace of X and $f \in Y^*$ such that $f(x) \leq p(x)$ for all $x \in Y$. Then, there exists $F \in X^*$ extending f to X such that $F(x) \leq p(x)$ for all $x \in X$.

   Verify that the defined p is in the hypothesis of Hahn-Banach lemma.

   Let F be the extention. Then $F(g) > 0$ for all $g \in G$. Notice that $p(-g) = 0$ for all $g \in G$, then if $F(g) < 0$, then $F(-g) > 0$ which contradicts $F(-g) \leq p(-g) = 0$. If $F(g) = 0$ then it is 0 on $-g$ as well, therefore on some neighborhood of $-g$, because C is open and consequently a neighborhood of g is in G. Consequently F is 0 everywhere. Then it is enough to set $H = \{x \in E \colon Fx = 0\}$.

7. Decide whether following functionals on a normed linear space X are linear and continuous. If so, determine their norm.

   (a) $F : (x_n)_{i \in \mathbb{Z}^+} \mapsto \sum_{i=1}^{\infty} \frac{x_i}{i^2}$, $X = c_0$

   Řešení: It is well defined, because $\sum_i 1/i^2 < \infty$. It is linear. Its norm is $\sum_i 1/i^2$.

   (b) $F : f \mapsto \int_0^1 t f(t)\,\mathrm{d}t$, $X = L^p([0, 1])$

   Řešení: It is linear. It is continuous:

$$\left|\int_0^1 t f(t)\,\mathrm{d}t\right| \leq \int_0^1 t\,|f(t)|\,\mathrm{d}t \leq \int_0^1 |f(t)|\,\mathrm{d}t = \|f\|_1 \leq \|f\|_p$$

   Computation of norm: for $p = 1$ the sequence of funtions ix^{i-1} show that the norm is 1. For $p = \infty$ take the function constant 1, so the norm is $1/2$.

   For the general case we will need Hölder inequality: for $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$ then $\|fg\|_1 \leq \|f\|_p \|g\|_q$. Let $f = t^{q/p}$, then

$$\|F\| = \left\|t^{q/p}\right\|_p \|t\|_q \big/ \left\|t^{q/p}\right\|_p = \sqrt[q]{\int_0^1 t^q} = \left(\frac{1}{1+q}\right)^{1/q} = \left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}}$$

(c) $F : f \mapsto \lim_{n\to\infty} \int_0^1 f(t^n)\,\mathrm{d}t,\ X = \mathcal{C}([0,1])$

Řešení: It is linear. It is continuous:

$$\left|\lim_{n\to\infty} \int_0^1 f(t^n)\,\mathrm{d}t\right| \leq \lim_{n\to\infty} \int_0^1 \max f\,\mathrm{d}t = \|f\|$$

Function $f = 1$ shows that the norm is 1.